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Theory of edges of leaves

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Abstract. – We performed experiments in which tearing pieces of plastic produced a fractal boundary. Similar patterns are commonly observed at the edges of leaves. These patterns can be reproduced by imposing metrics upon thin sheets. We present an energy functional that provides a numerical test-bed for this idea, and derive a continuum theory from it. We find ordinary differential equations that provide minimum energy solutions for long thin strips with linear gradients in metric, and verify both numerically and experimentally the correctness of the solutions.

Introduction. – If one takes a thin piece of plastic, such as a garbage bag, and rips it in half, the edge takes on a complicated rippled form. We recently performed a careful experimental study of torn sheets and found the edge can have a fractal character, with waves superimposed upon waves over many generations [1]. Our aim in this letter is to provide some explanations for the waves appearing in these systems.

We do not believe that details of how plastic tears are important, and we will not discuss them. All that appears to be essential is that the act of tearing imposes a new metric upon the thin sheet of plastic, one in which the plastic is uniformly elongated as one moves toward the newly formed edge. Similar elongation can be created by completely different physical processes, such as growth laws in plants. Therefore, it is not surprising that the edges of many leaves and flowers exhibit very similar shapes. We present in fig. 1 a comparison between torn plastic sheets, leaves, and some numerical experiments that motivated our studies.

Energy of a thin sheet. – The energy of a thin sheet is conventionally given by the Föppl-von Kármán equations, and is the sum of two terms, one involving bending of the sheet and the other involving stretching [2, 3]. We start instead with a physical model of a sheet that is based upon a discrete collection of interacting points, and allows us to move easily back and forth between numerical and analytical approaches. We derive the continuum theory from our discrete model rather than by discretizing continuum equations. Similar numerical techniques have been employed frequently in studies of crumpled paper and tethered membranes [4, 5].

Let $\vec{u}_i$ be a collection of mass points that interact with neighbors, and at rest form a thin flat sheet. Let $\vec{\Delta}_{ij}$ be equilibrium vector displacements between neighbors i and j. When

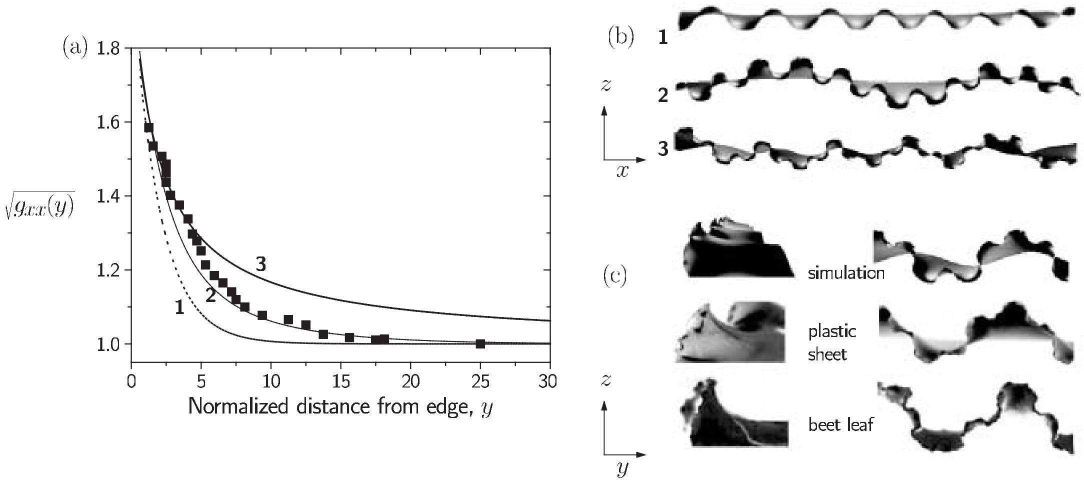

Fig. 1 – The relation between the buckling cascade and the metric of the sheet. (a) The experimentally measured metric function $g_{xx}(y)$ in a 0.2 mm thick sheet (squares) is compared with three different metrics used in numerical minimizations of eq. (1): 1) $g_{xx}(y) = (1 + \exp[-y/2])^2$ (dotted line), 2) $g_{xx}(y) = (1 + 0.7\exp[-y/2] + 0.3\exp[-y/6])^2$ (thin line) and 3) $g_{xx}(y) = (1 + 2/(2 + y))^2$ (bold line), where y is in units of sheet thickness. (b) Minimum energy configurations of the energy $\mathcal{E}$ in eq. (1) for the three functional forms in (a): 1 (top), 2 (middle) and 3 (bottom). (c) Side (y-z plane, right) and front (x-z plane, left) views from simulation (metric as in (a3)), experiment, and beet leaf.

the neighbors are not in equilibrium, the distance between them is $u_{ij} = |\vec{u}_j - \vec{u}_i|$. Let $g_{\alpha\beta}$ be a metric tensor that describes permanent deformations of the sheet away from its original equilibrium. We begin with an energy functional that makes physical sense and leads to a familiar continuum limit:

$$\mathcal{E} = \frac{\mathcal{K}}{2a} \sum_{\langle ij \rangle} \left[u_{ij}^2 - \sum_{\alpha\beta} \Delta_{ij}^{\alpha} g_{\alpha\beta} \Delta_{ij}^{\beta} \right]^2, \tag{1}$$

where $\mathcal{K}$ has dimensions of energy per volume, and a has dimensions of length. All that has happened, in short, is that the equilibrium distance between material points has changed. Our hypothesis, also adopted in a closely related problem by Nechaev and Voituriez [6], is that by specifying different metric tensors g, often ones with very simple functional forms, we can describe all the buckling cascades we have observed in the laboratory. In particular, since the plastic sheets are torn uniformly in the x-direction, we will assume that the metrics g are constant in the x-direction, and vary only along y. We take g to be diagonal, with $g_{xx}(y)$ a smoothly varying function and $g_{yy} = 1$. Suggestive results from numerical experiments where we guessed various metrics and minimized eq. (1) appear in fig. 1.

Continuum limit. – Analytical progress depends upon moving to the continuum limit of eq. (1). Define the Cauchy-Green strain tensor

$$\epsilon_{\alpha\beta} \equiv \frac{1}{2} \left[\sum_{\gamma} \frac{\partial u^{\gamma}}{\partial r_{\alpha}} \frac{\partial u^{\gamma}}{\partial r_{\beta}} - g_{\alpha\beta} \right]. \tag{2}$$

Then in the continuum limit $\mathcal{E}$ can be rewritten as

$$\mathcal{E} = \frac{\mathcal{K}}{a} \sum_i \sum_{j\,\mathrm{nbr.\,of}\ i} \left[\sum_{\alpha\beta} \Delta^{\alpha}_{ij} \epsilon_{\alpha\beta}(\vec{r}_i) \Delta^{\beta}_{ij} \right]^2 . \tag{3}$$

One has to specify a particular lattice in order to proceed further. If one takes the points $\vec{u}_i$ to sit on a two-layer system of triangular lattices, stacked over one another as in the first stage of forming an hcp lattice of lattice constant a, the energy takes the particular form

$$\mathcal{E} = \frac{\mathcal{K}a^3}{24} \sum_i \begin{bmatrix} & 19\big[\epsilon_{xx} + \epsilon_{yy}\big]^2 + 38\big[\epsilon^2_{xx} + \epsilon^2_{yy} + 2\epsilon^2_{xy}\big] \\ + & 32\epsilon^2_{zz} + 16\big[\epsilon_{yy} + \epsilon_{xx}\big]\epsilon_{zz} + 32\big[\epsilon^2_{yz} + \epsilon^2_{xz}\big] \\ + & 8\sqrt{2}\big[\epsilon_{yz}(\epsilon_{yy} - \epsilon_{xx}) + 2\epsilon_{xy}\epsilon_{xz}\big] \end{bmatrix} . \tag{4}$$

Solvable problem at edge of strip. – The full problem at hand is to find the minimum energy state of eq. (1) or eq. (4) for a very long strip of finite width w and very small thickness t, subject to a metric $g_{xx}(y)$, where g_{xx} has some value greater than 1 at $y = 0$ (fig. 1(a)), and decreases monotonically toward 1 as y approaches the other side of the strip. It seems very unlikely that this problem has in general an exact analytical solution.

However, we have found a particular case where the problem can largely be solved [7]. The motivation for the solvable problem comes by looking at the left panels of fig. 1(c), and imagining making a new strip by slicing off the material at distance $y = w$ from the left-hand side. If w is small enough, then the metric g_{xx} within it should have the form of constant plus a term linear in y. Therefore, we study a thin strip whose metric is

$$\sqrt{g_{xx}(y)} = (1 - y/R), \qquad g_{yy} = 1, \tag{5}$$

where R is a constant and $y \in [-w/2, w/2]$. We look for solutions $\vec{u}(x, y, z)$ subject to the constraint that there be some strip length L and period λ for which

$$\vec{u}(x + L, y, z) = \hat{x}\lambda + \vec{u}(x, y, z). \tag{6}$$

Studying thin strips with linear metric gradients does not permit one to analyze all the complex behavior appearing in fig. 1, but it does allow one to calculate a basic undulating shape that appears as an ingredient of the buckling cascade. To form a rough correspondence between the two problems, choose $L/\lambda \equiv \sqrt{g_0}$ to be the amount the membrane in fig. 1(c) has been stretched at the left edge.

If a strip of length L is given metric (5) and no constraint is applied, then the minimum energy configuration is easy to find. The material can relax completely by forming a ring of radius R, which curls round and round in a circle. Thus one can obtain some intuition about the problem by cutting out the paper figure in fig. 2 and pulling the ends apart. Let $\hat{r}_1(\theta)$ and $\hat{r}_2(\theta)$ be two unit vectors attached to the circular strip, where $\hat{r}_1$ points around the circumference, and $\hat{r}_2$ points along the radius. Then the lowest-energy deformations of the paper strip are described by two functions $\dot{\psi}(\theta)$ and $\dot{\phi}(\theta)$ that describe the rate of rotation around axes $\hat{r}_1$ (twisting) and $\hat{r}_2$ (bending), respectively. The directions of the unit vectors are determined by

$$\frac{\partial \hat{r}_1}{\partial \theta} = \dot{\phi}(\theta)\, \hat{r}_3 + \hat{r}_2, \tag{7a}$$

$$\frac{\partial \hat{r}_2}{\partial \theta} = \dot{\psi}(\theta)\, \hat{r}_3 - \hat{r}_1, \tag{7b}$$

$$\hat{r}_3 = \hat{r}_1 \times \hat{r}_2. \tag{7c}$$

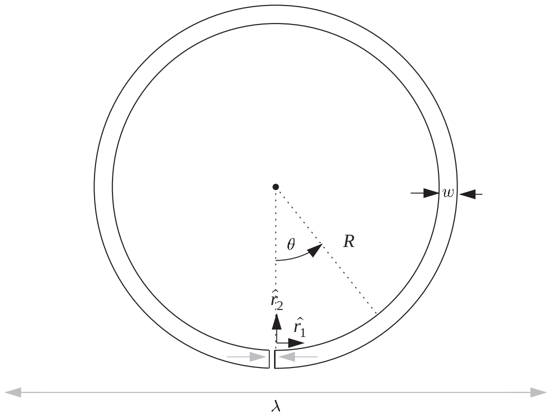

Fig. 2 – The buckled shapes studied in this paper can be created by cutting out this circular strip, of radius R, length $L = 2\pi R$, and width w. Rotate the two ends around $\hat{r}_1 \times \hat{r}_2$ so that the two gray arrows point 180° away from one another, and place them over the lower line so that the arrows coincide. The paper will assume the undulating shape from fig. 3 with $\sqrt{g_0} = 2$. Moving the ends to other horizontal distances λ along the line, or choosing other points along the strip for arclength L, allows one to explore a two-parameter family of periodic strip solutions.

It is easy to check that $\hat{r}_1$, $\hat{r}_2$, and $\hat{r}_3$ remain orthonormal unit vectors under the dynamics described by eqs. (7). One way to derive these equations is to consider the evolution of three orthonormal vectors where the rotation rate of $\hat{r}_1$ and $\hat{r}_2$ around $\hat{r}_3$ is constrained to be unity; this constraint comes from the fact that the strip width w is much wider than the thickness t and any additional rotation of $\hat{r}_1$ and $\hat{r}_2$ around $\hat{r}_3$ would stretch the strip. The center-line $\vec{l}(\theta)$ of the strip is given by

$$\vec{l}(\theta) \equiv R \int_0^\theta \mathrm{d}\theta' \hat{r}_1(\theta'). \tag{8}$$

Once the unit vectors $\hat{r}_1(\theta)$, $\hat{r}_2(\theta)$, and $\hat{r}_3(\theta)$ have been determined, then the displacements of all points in the strip are given by

$$\vec{u}(R\theta, y, z) = \vec{l}(\theta) + \hat{r}_2 y + \hat{r}_3 z + \sum_{i=1}^{3} \left[q_{zz}^{(i)} z^2 + q_{zy}^{(i)} zy + q_{yy}^{(i)} y^2\right] \hat{r}_i / R. \tag{9}$$

The quantities $q_{\alpha\beta}^{(i)}$ are functions of θ, and are needed to allow for Poisson contractions of the strip along the y and z directions. They can be determined by inserting eq. (9) into eq. (4) and differentiating to find a local minimum, giving

$$\begin{aligned} q_{zz}^{(1)} &= \sqrt{2}\dot{\psi}/4, \qquad q_{zy}^{(1)} = -\dot{\psi}, \qquad q_{yy}^{(1)} = 0,\\ q_{zz}^{(2)} &= \sqrt{2}\dot{\phi}/6, \qquad q_{zy}^{(2)} = \dot{\phi}/3, \qquad q_{yy}^{(2)} = 0,\\ q_{zz}^{(3)} &= \dot{\phi}/12, \qquad q_{zy}^{(3)} = 0, \qquad q_{yy}^{(3)} = -\dot{\phi}/6. \end{aligned} \tag{10}$$

Making use of these results, one can finally find the energy $\mathcal{E}$ needed to bend a strip into a shape determined by the bending and twisting rates $\dot{\phi}$ and $\dot{\psi}$, and it is

$$\mathcal{E} = \mathcal{K} \frac{2a^3 w}{3\sqrt{3}R} \int_0^{L/R} \mathrm{d}\theta \left(C_1 \dot{\phi}^2 + C_2 \dot{\psi}^2\right), \quad \text{with } C_1 = 2,\ C_2 = 3. \tag{11}$$

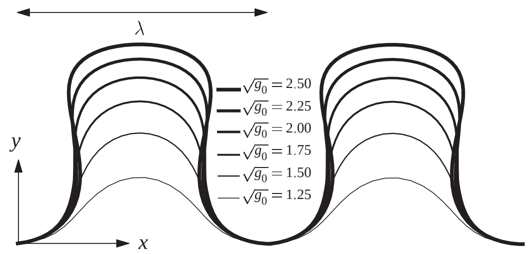

Fig. 3 – Analytical solutions of eqs. (13) with $C_1 = C_2$. The solutions are depicted for constant wavelength λ, and varying strip length $L = \sqrt{g_0}\lambda$.

Actually, this expression describes the low-energy states of a strip only in a certain limit. The limit is

$$\left(\frac{w}{R}\right)^2 \ll \frac{t}{R} \ll \frac{w}{R}, \tag{12}$$

which means that the strip is much wider than it is thick, but not so wide that it becomes favorable to develop structure across the width of the strip.

Euler-Lagrange equations. – There still remains the task of finding the minimum energy shape of the strip. This problem was recently solved by Audoly and Boudaoud [8]. They parameterized the solution with Euler angles, and the expressions are rather lengthy. We present below an alternative solution that is more compact. One must determine the bending and twisting rates $\dot{\phi}$ and $\dot{\psi}$, subject to the boundary conditions implied by eq. (6). We proceed by rewriting eq. (11) in terms of derivatives of $\hat{r}_1$ and $\hat{r}_2$, using Lagrange multipliers to enforce the kinematic constraints in eqs. (7). Extremizing the resulting functional with respect to $\hat{r}_1$ and $\hat{r}_2$ and eliminating most of the Lagrange multipliers gives

$$\frac{\partial\dot{\psi}}{\partial\theta} = h\dot{\phi}/C_2 + \dot{\phi}, \tag{13a}$$

$$\frac{\partial\dot{\phi}}{\partial\theta} = \left[p\hat{x}\cdot\hat{r}_3 - h\dot{\psi}\right]/C_1 - \dot{\psi}, \tag{13b}$$

$$\frac{\partial h}{\partial\theta} = p\hat{x}\cdot\hat{r}_2 + [C_1 - C_2]\dot{\phi}\dot{\psi}. \tag{13c}$$

Here p is a constant that can be varied to try to match boundary conditions. Together with eqs. (7), eqs. (13) completely specify the problem. A set of equations with the same physical content is found in [8].

The solutions of eqs. (13) are indexed by four constants: the initial values $\dot{\phi}(0)$ and $h(0)$, the normalized strip length L/R, and p. The initial value $\dot{\psi}(0)$ vanishes because $\dot{\psi}(\theta)$ is odd. By varying $\dot{\phi}(0)$ and p, it is possible to ensure that $\hat{r}_1(L/R) = \hat{x}$ and $\hat{r}_2(L/R) = \hat{y}$. The remaining constant $h(0)$ can then be employed to vary the wavelength λ. Thus the solutions can be indexed by λ and L, which can be varied independently within certain ranges. To observe these solutions, one can cut out the strip in fig. 2, pick various strip lengths L, and force the strip at these points to lie at distance λ from the origin.

Closed-form solution. – In a special case, eqs. (7) and (13) have a closed-form solution [7]. Set $C_1 = C_2$, and take

$$\dot{\phi} = \omega\cos\alpha\theta, \qquad \dot{\psi} = \omega\sin\alpha\theta. \tag{14}$$

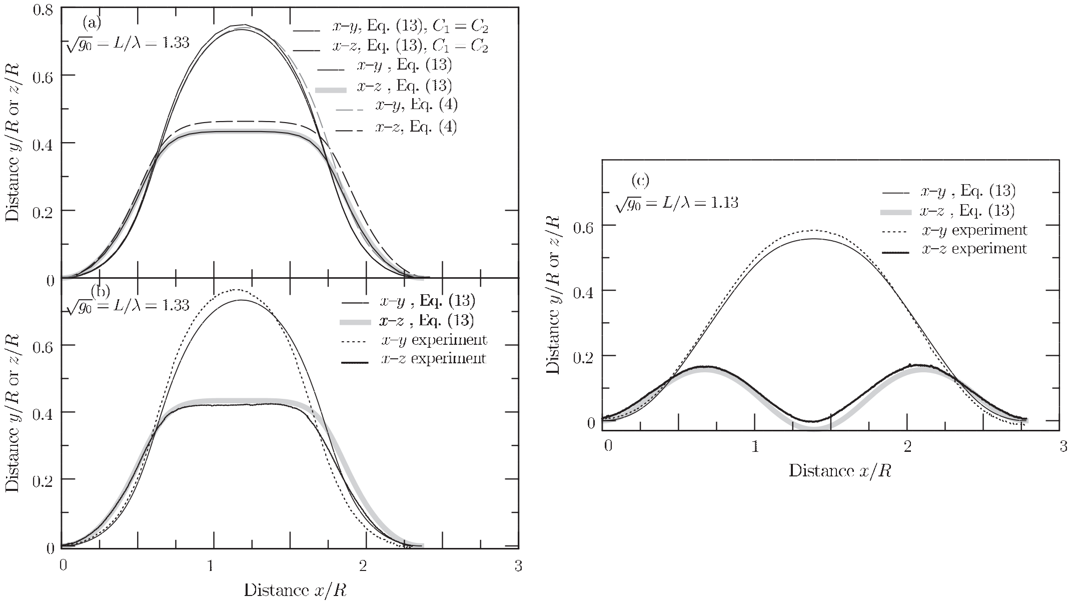

Fig. 4 – Theory and experiment for shapes of strips. (a) Comparison of three separate methods to find the shape of a strip. The first results from solving eqs. (13) by a numerical shooting technique for $L/\lambda = \sqrt{g_0} = 4/3$, and $L/R = \pi$. The second results from setting $C_1 = C_2$, permitting a solution in terms of elementary functions. The third method is a direct numerical minimization of eq. (1) for a strip 201 atoms long in the x-direction, and 12 layers wide in the y-direction, given curvature $R = 63.7$, constrained to have horizontal period of 150, so that again $\sqrt{g_0} = 4/3$. The true minimum energy from eqs. (13) and (11) is $1.3557\mathcal{K}a^3$. The energy predicted by eq. (11) with the approximate forms for $\dot{\phi}$ and $\dot{\psi}$ from eq. (14) is $1.3571\mathcal{K}a^3$. The energy found from direct numerical minimization of eq. (1) is $1.377\mathcal{K}a^3$. (b) Strips of polycarbonate are cut out with width $w = 0.164\,\text{cm}$, radius $R = 2.662\,\text{cm}$ and extending through length $L/R = \pi$. Ends of the strip are held such that $\sqrt{g_0} = L/\lambda = 4/3$, and the results are compared with the corresponding solution of eqs. (13). (c) Ends of the strip are held such that $\sqrt{g_0} = L/\lambda = 1.13$ and are compared with the corresponding solution of eqs. (13).

For given $L/\lambda = \sqrt{g_0}$, there is a special wavelength

$$\lambda = \frac{4\pi R(\sqrt{g_0} - 1)}{g_0} \tag{15}$$

for which $\vec{u}$ can be expressed purely in terms of trigonometric functions. Graphs of these solutions appear in fig. 3. Although they are exact only when $C_1 = C_2$, as shown in fig. 4(a), they lie extremely close to a numerical solution of eqs. (13) with $C_1 = 2C_2/3$ for the same boundary conditions. In addition, these closed-form solutions provide excellent starting points when one searches numerically for more general solutions. The special wavelength λ in eq. (15) is only special because it makes analytical solution possible, and not because it provides an energy minimum.

Numerics and experiments. – We have carried out a direct numerical minimization of eq. (1), and compared both the shape and energy of the resulting solution with the analytical predictions. The results are displayed in fig. 4(a). The closed-form solution available when

$C_1 = C_2$ is almost indistinguishable from the numerical solution of eqs. (13) when $C_1 = 2C_2/3$. The direct minimization of eq. (1) deviates slightly from the exact solution, probably because of incomplete convergence and finite-size effects in the numerics.

As a final check of our understanding, we cut strips similar to those in fig. 2 from sheets of polycarbonate, pinned their ends as described in the caption, and used a scanning profilometer to find the shape of their centerlines. In figs. 4(b) and (c) we compare solutions of eqs. (13) to the measured shapes. Note that once L/R and λ/R are determined, the shape is completely specified and there are no free parameters. Agreement between predicted and experimental shapes is satisfactory

It is possible that the lowest-energy state for the simple metric in eq. (5) involves a cascade of oscillations on many scales, but we have not yet obtained detailed evidence. Many other problems remain to be addressed, including the application of the ideas obtained here to the more complex metrics illustrated in fig. 1 for torn sheets and leaves.

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