Density variations in a one-dimensional granular system

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In this work we examine a system of inelastic particles confined to move on a line between an elastic wall and a heat source. Solving a Boltzmann equation for this system leads to an analytic expression for steady state behavior. Numerical simulations show that the system is in fact capable of simultaneously displaying both the uniform density of the analytic solution, and a state in which the particles are collected into a cluster adjacent to the elastic wall. The boundary conditions for the Boltzmann treatment are then reworked to provide a theoretical description of how smooth particle distributions and clumping phenomena can coexist. From this, we gain a prediction for the time scale of clump formation in this system. © *1996 American Institute of Physics.*
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I. INTRODUCTION

The spontaneous creation of large scale structure in an initially homogeneous system is a recurring phenomenon in physics. Granular systems offer some unusual examples of this behavior. Despite the absence of long range forces between the particles, large variations in density still exist. In two-dimensional systems a non-uniform cooling process has been observed.[1,2] Regions of dense, slow particles spontaneously develop, with a few higher velocity particles moving quickly through the voids. These variations in density and speed occur regardless of the smoothness of the initial conditions. Similar phenomena have also been seen in one dimension.[3,4]

In this work, a system of inelastic particles on a line is used to study the mechanisms involved in density fluctuations. So as to create steady state behavior, the system has an energy source to balance the dissipation due to collisions. Even in a non-cooling system, we see density and energy variations: a state composed of several rapidly moving particles and one relatively stationary clump. If the coefficient of restitution is r, then the size of this clump is of the order of $(1-r)$ times the number of particles in the clump, while the average energy within the clump is of the order of $(1-r)$ times the average energy of the particles in motion. The grouping of particles in a driven one-dimensional system has been observed previously,[5] but here we see the coexistence of the practically stationary clump and many high velocity particles. In this paper we use a Boltzmann treatment to obtain a partial differential equation describing the distribution function for the particles in the system. This equation is solved analytically for the case of elastic particles and no clump. The quasi-elastic problem is then treated as a perturbation to this solution. The clump can then be perceived as an alteration to the boundary conditions for the Boltzmann equation, and the methods developed for the no clump case can be used to calculate an expression for the steady state distribution function of the moving particles when the system includes a clump. Simulations were used to examine the mechanisms of clump formation; these considerations suggest an analytic description of the process that allows us to predict the time scale of the clump formation.

It has been shown previously[6–9] that particles undergoing sufficiently inelastic collisions can dissipate all their energy in the center of momentum frame within a finite amount of time. This process has been termed ''inelastic collapse,''[7] and it requires an infinite number of collisions during which the particles' relative separations and velocities go to zero so that the particles come into contact. In order to ensure that the system is not in this regime, the situations described in this work are limited to those in which the system is quasi-elastic, i.e., there are not enough particles in the box to form the collapse singularity. Our density fluctuations are distinct from the cluster formed in inelastic collapse because the internal energy and size of the clump do not vanish in a finite time.

II. THE MODEL

In this work, we examine the behavior of N identical particles confined to move on the line between $x=0$ and $x=1$. At $x=1$, there is an elastic wall, i.e., when a particle of velocity v hits this wall it is reflected with a velocity $\overline{v}=-v$. The collisions between particles are inelastic: they conserve momentum, but not energy. The degree of inelasticity is parameterized by the coefficient of restitution, r. When two particles with speeds v_1 and v_2 collide, their new relative velocity is just $-r$ times their old relative velocity:

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$\overline{v_2}-\overline{v_1}=-r(v_2-v_1)$. Using this and conservation of momentum, we see that their final velocities $\overline{v_1}$ and $\overline{v_2}$ can be written as

$$\overline{v_1}=qv_1+pv_2, \quad \overline{v_2}=pv_1+qv_2, \tag{1}$$

where we have defined

$$q=(1-r)/2 \quad \text{and} \quad p=(1+r)/2. \tag{2}$$

Thus $r=1$ is the elastic case, whereas $r=0$ corresponds to total inelasticity (particles collide and move together). In this paper we examine systems in the quasi-elastic regime (r very near one), so we will characterize the degree of inelasticity by $q \ll 1$ and do most calculations to first order in this variable.

Since the particles are colliding inelastically, the system loses energy $[\Delta E=-\frac{1}{4}(1-r^2)(v_2-v_1)^2]$ at each collision. Thus, in the center of momentum frame, the particles are all decelerating toward zero velocity. In order to look at steady states of this system, it is necessary to provide a forcing mechanism that pumps energy back into the system. One possibility is to put the particles above a vibrating plate in a gravitational field (as in Refs. 8,10 and references therein). Another option is a vibrating horizontal box: the particles that hit a moving wall can gain energy from it. This model was first proposed with one particle, by Fermi,[11] as he tried to understand cosmic radiation, and it became a classical example in the theory of dynamical systems.[12] A drawback of both these models is that they involve periodic motion of the wall, and hence the particles can get phase-locked and trapped in a periodic state.[13] Similar resonances have recently been observed in two dimensions by McNamara and Barrat.[14] To separate the effects of phase-locking and resonance from effects that are intrinsic to the inelastic nature of granular systems, we will focus on the idealized thermal energy source proposed in Ref. 5. Particles hit the right wall ($x=1$) and bounce off elastically. When a particle hits the left wall ($x=0$), it picks a random speed $v>0$, from the one-sided distribution $W(v)$ with $\int_0^\infty W(v)dv=1$. The outgoing velocity (always positive) is uncorrelated with the incoming velocity (always negative). In this work we will often use the family of density functions,

$$W_\alpha(v)=2^{(1-\alpha)/2}v^\alpha e^{-v^2/2}H(v)/\Gamma((\alpha+1)/2), \tag{3}$$

where $H(v)$ is the Heaviside function and $\Gamma(n)\equiv\int_0^\infty y^{n-1}e^{-y}dy$ is the gamma function. Here α describes both the strength of the forcing [the average energy of a particle leaving the wall is $(\alpha+1)/2$] and the behavior of the distribution function near the origin [$W_\alpha(v) \propto v^\alpha$ for small v]. Later we will see that the latter property plays a large part in determining the long term behavior of the system. Although most of the calculations are valid for any distribution $W(v)$, this family contains some of the more interesting cases, including the Gaussian distribution (see Fig. 1).

This boundary condition is neither a constant temperature nor a constant flux condition. Indeed, the amount of energy transferred to the system depends on the properties of the incoming particles. To compute the energy injected into the medium by the wall in a unit time, one has to sum the energy of the all particles leaving the wall and then subtract the energy those particles had when they hit the wall. Thus the net energy flux supplied by the wall is a function of the velocities of the particles coming into the wall. The same argument applies when we try to calculate the temperature at the wall. Nonetheless, this model for boundary forcing does provide a simple way of idealizing the energy injection process.

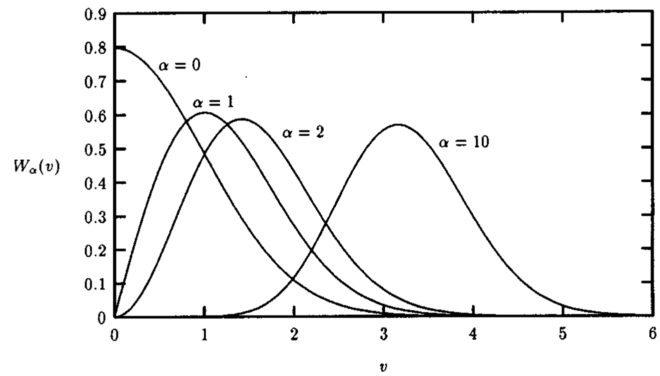

FIG. 1. Several examples taken from the family $W_\alpha(v)$ defined in (3). When α increases, $W_\alpha(v)$ becomes flatter around $v=0$.

Notice also that this boundary condition acts as a source of randomness in the system. Recent studies[4,16] have shown the spontaneous development of correlations in speed and position of inelastic particles in one dimension. When the particles hit the wall, these correlations will be reduced due to the ''loss of memory'' character of the boundary condition.

III. BOLTZMANN EQUATION

We assume that the particles do not clump or cluster together, and thus are non-correlated so we can use statistical tools. Define the phase space density function $f(x,v,t)$ to be such that the number of particles at time t, between x and $x+dx$, with velocity between v and $v+dv$, is $f(x,v,t)dx\, dv$. $f(x,v,t)$ is governed by a one-dimensional Boltzmann equation (see Ref. 15), which describes the conservation of particles. In Ref. 16, it is shown that if r is close to 1, the Boltzmann equation takes the form

$$f_t+vf_x+(af)_v=0, \tag{4}$$

where $q=(1-r)/2$ and

$$a(x,v,t)\equiv q\int_{-\infty}^{\infty}|v'-v|(v'-v)f(x,v',t)dv' \tag{5}$$

is the acceleration of a particle at (x,v) in the phase space.

A physical derivation of this equation might be instructive: imagine a test particle with speed v at position x, moving through a cloud of all the other particles. If the system were elastic, each collision would merely result in an exchange of velocities. Thus, when the particles are relabeled appropriately, it can be seen that the system is unchanged. In our quasi-elastic system, the velocities are almost, but not exactly, exchanged. We can compute the effect of each collision and hence the test particle's average acceleration.

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• First consider the collision of the test particle with a particle of speed v'. After this collision, the particles are relabeled. The particle originally with velocity v' is now the test particle, as, from (1), its final velocity is $pv+qv'$ which, in the quasi-elastic limit, is close to v. Thus the modification of the test particle's velocity is $\Delta v/\text{collision} = (pv+qv') - v = q(v'-v)$.

• The test particle encounters ΔN particles with speed between v' and $v'+dv'$ in a time Δt, where $\Delta N = |v'-v| f(x,v',t) dv' \, \Delta t$. The acceleration of the test particle due to these encounters is then $da = \Delta v/\Delta t = (\Delta N/\Delta t)(\Delta v/\text{ collision}) = q(v'-v)|v'-v| f(x,v',t) dv'$.

• Now integrate over every v' to find the average total acceleration of the test particle due to all the other particles: $a = q\int_{-\infty}^{\infty}(v'-v)|v'-v| f(x,v',t) dv'$.

The energy is supplied through interactions with the wall and therefore the boundary conditions at $x=0$ and 1 are essential in this calculation. The right-hand wall at $x=1$ is elastic, and hence

$$f(1,v,t) = f(1,-v,t). \tag{6}$$

The main complication comes from the energy source at the left wall, $x=0$. During a time interval dt, there are a number, dN, of particles that leave this wall with velocities between v and $v+dv$ (where $v>0$). These outgoing particles must have been produced from the number, dN', of incoming particles that arrived at the wall in this time with any velocity $v'<0$. If we look at the system at a given time t, the dN particles that have left the wall in the past dt are now spread out between $x=0$ and $x=v\ dt$, while if we had looked at $t-dt$, the dN' arriving particles would have been between $x=0$ and $x=-v't$. Thus

$$dN = f(0,v,t) dv (v\ dt),$$

$$dN' = \int_{-\infty}^{0} |v'| f(0,v',t) dv'.$$

The probability of any impinging particle (one of dN') being ejected by the $x=0$ wall with a velocity between v and $v+dv$ is given by $W(v)dv$. Thus $dN = dN' W(v) dv$, or

$$v f(0,v,t) = W(v) R(t), \quad \forall v>0, \tag{7a}$$

$$R(t) \equiv \int_{-\infty}^{0} |v'| f(0,v',t) dv'. \tag{7b}$$

Notice that $R(t)>0$ is the rate at which particles hit the left-hand wall.

The final condition on $f(x,v,t)$ is normalization; the number of particles in the system is fixed at N:

$$N = \int_{-\infty}^{\infty}\int_{0}^{1} f(x,v,t) dx\ dv. \tag{8}$$

It is easy to verify that (4) with the boundary conditions (6) and (7) conserves N.

IV. ELASTIC BOLTZMANN EQUATION

A first step in understanding the quasi-elastic model is to study the simpler case, $r=1$. This is a perfectly elastic one-dimensional gas forced by the boundary conditions (6) and (7). In one dimension, the elastic collision rule (1) is equivalent to an exchange of velocities, thus we can treat the system as a collection of non-interacting particles.

A. Steady states of the perfect gas

In this section we will study the possible steady states of the perfect gas. Since $r=1$ yields $q=(1-r)/2=0$, (4) becomes the one-dimensional elastic Boltzmann equation,

$$f_t + v f_x = 0. \tag{9}$$

A steady state must satisfy the even simpler equation

$$v f_x = 0,$$

which is easily integrated to find $f(x,v) = C(x)\delta(v) + F(v)$. We are interested in the effect of the energy source at $x=0$, so let $C(x)=0$, i.e., ignore the solutions that include a bunch of particles at rest. The actual form of $F(v)$ is determined by the boundary conditions: Equation (6) implies that $F(v)$ is an even function, and Equation (7a) gives that $F(v)$ is proportional to $W(|v|)/|v|$, since, for the steady state problem, $R(t)$ is no longer dependent on time. The normalization condition (8) becomes

$$N = \int_{-\infty}^{\infty} F(v') dv' = 2R\int_{0}^{\infty}\frac{W(v')}{v'} dv', \tag{10}$$

which allows us to calculate R, the rate of collisions with the wall at $x=0$. The integral in (10) is infinite if $W(0) \neq 0$. If, for example, $W(v) = W_\alpha(v)$, the form described in (3), then Equation (10) shows that there is no steady solution for $\alpha=0$. When $\alpha \neq 0$, R is well defined by (10), and the steady state distribution function is

$$f(x,v) = \frac{NW(|v|)}{2|v|\int_0^\infty v'^{-1} W(v') dv'} \equiv N\phi(v). \tag{11}$$

Notice that because of the factor $|v|$ in the denominator of (11) the velocity distribution function of the wall, $W(v)$, is not imposed on the medium. This can be understood physically by following one particle. At each interaction with the left wall, it picks a velocity with probability $W(v)$. However, it keeps this velocity during a time $\Delta t = 2/|v|$ (the time it takes to travel back and forth in the box). Therefore a time average should use a probability distribution proportional to $W(|v|)/|v|$. The ergodic assumption turns averaging over time for one particle into averaging over the ensemble of particles. Therefore the probability distribution function should be proportional to $W(|v|)/|v|$. When $\alpha=0$, this function is not integrable: there cannot be a steady state.

B. Time dependent solution for the elastic model

In order to study what happens if $\alpha=0$, we will now solve (9), and compute the evolution of $f(x,v,t)$ with time. Let $f(x,v,0) = f_0(v)$ be the initial distribution, where $f_0(-v) = f_0(v)$. Equation (9) can be integrated using the method of characteristics with

$$f(x,v,t) = F(x-vt,v),$$

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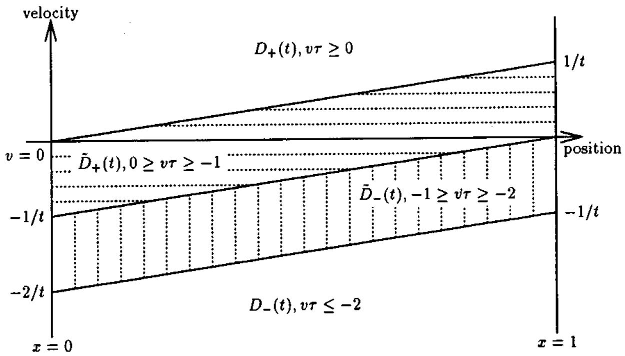

FIG. 2. In order to solve the elastic Boltzmann equation, the phase space is divided into four time dependent regions. Notice that, as $t\rightarrow\infty$, the two shaded areas, $\widetilde{D}_+(t)$ and $\widetilde{D}_-(t)$, shrink towards the line $v=0$.

where F is determined using boundary conditions. Note that, since the system is elastic, a collision between two particles is equivalent to an exchange of velocities. Thus the distribution function is affected only by collisions with the wall.

At a particular time t, there are four distinct domains in the (x,v) phase space, as shown in Fig. 2.

- For a particle with $v>0$ and $0\leqslant t-x/v\equiv\tau$, the last collision with a wall occurred at $x=0$. The boundary condition there (7a) yields

$$f(x,v,t)=f(0,v,\tau)=R(\tau)W(v)v^{-1}, \tag{12a}$$

which holds when $v\tau\geqslant 0$. Call this domain $D_+(t)$.

- The next region contains particles which are moving so slowly they have not yet hit any wall [so either $v>0$ and $0>\tau$ or $v<0$ and $0\geqslant t-(1-x)/(-v)=\tau+1/v$]. These particles still have the initial velocity distribution:

$$f(x,v,t)=f(x-vt,v,0)=f_0(v), \tag{12b}$$

which holds when $0>v\tau\geqslant -1$. Call this domain $\widetilde{D}_+(t)$.

- This region includes all particles which have collided only with the right-hand wall, so $v<0$ and $0<t-(1-x)/(-v)=\tau+1/v$ but $0\geqslant t-(2-x)/(-v) = \tau+2/v$. Using the boundary condition (6) at $x=1$, the previous result (12b), and the evenness of f_0, we find

$$f(x,v,t)=f(1,v,\tau+1/v)=f(1,-v,\tau+1/v)=f_0(v), \tag{12c}$$

which holds when $-1>v\tau\geqslant -2$. Call this domain $\widetilde{D}_-(t)$.

- The last region is for those particles that have collided with the left-hand wall and then with the right-hand wall, so $v<0$ and $0<t-(2-x)/(-v)=\tau+2/v$. Using the results from $\widetilde{D}_-(t)$ and $D_+(t)$, we find

$$\begin{aligned}f(x,v,t)&=f(1,v,\tau+1/v)\\&=f(1,-v,\tau+1/v)=W(|v|)|v|^{-1}R(\tau+2/v),\end{aligned} \tag{12d}$$

which holds when $-2>v\tau$. Call this domain $D_-(t)$.

Notice that the boundary conditions at $x=0$ and 1 are satisfied by these solutions. It is also important to see that regions $\widetilde{D}_+(t)$ and $\widetilde{D}_-(t)$ are both shrinking to zero as time goes on (eventually all the particles will have collided with the left-hand wall), whereas as $t\rightarrow\infty$, $D_+(t)$ and $D_-(t)$ tend to cover the whole phase space (except the line $v=0$ whose measure is zero and therefore plays no role in our problem).

The expressions for $f(x,v,t)$ in the four regions depend only on the unknown function $R(t)$. Therefore using Equations (12) and the definition of $R(t)$ in (7b) we find:

$$R(t)=\int_0^{2/t}v'f_0(v')dv'+\int_{2/t}^{\infty}W(v')R\left(t-\frac{2}{v'}\right)dv'. \tag{13}$$

This integral equation (13) cannot be solved directly, but with the Laplace transform, $\widetilde{R}(p)=\mathscr{L}\{R(t)\} = \int_0^\infty e^{-pt}R(t)dt$, it takes the explicit form

$$\widetilde{R}(p)=\frac{\mathscr{N}(p)}{\mathscr{D}(p)}, \tag{14}$$

where

$$\mathscr{N}(p)\equiv\frac{1}{p}\int_0^\infty(1-e^{-2p/v'})v'f_0(v')dv', \tag{15a}$$

$$\mathscr{D}(p)\equiv 1-\int_0^\infty W(v')e^{-2p/v'}\,dv'. \tag{15b}$$

This set of equations, although explicit, cannot be used to find the exact expression for $R(t)$, due to the complexity of the inverse Laplace transform. However the behavior of $R(t)$ at large times can be seen from the structure of $\widetilde{R}(p)$ when $p\ll 1$. We therefore compute the behavior of both $\mathscr{N}(p)$ and $\mathscr{D}(p)$ as $p\rightarrow 0$. An easy computation leads to

$$\mathscr{N}(p)\approx 2\int_0^\infty f_0(v')dv'=N, \tag{16}$$

by the normalization condition. The expansion of $\mathscr{D}(p)$, however, depends sensitively on the structure of $W(v)$ for small v, and therefore on α.

1. Steady solution of the elastic model when $\alpha>0$

In the first part of Section III, we found a steady state solution (11) when $\alpha>0$. Calculating the time dependence in this case demonstrates that the system can evolve into this steady state at large times. For $\alpha>0$, $W(0)=0$ and $\mathscr{D}(p)$ is easily computed for small p:

$$\mathscr{D}(p)\approx 2p\int_0^\infty\frac{W(v')}{v'}dv'. \tag{17}$$

Therefore, as $t\rightarrow\infty$,

$$R(t)=\mathscr{L}^{-1}\left\{\frac{\mathscr{N}(p)}{\mathscr{D}(p)}\right\}\approx\frac{N}{2\int_0^\infty v'^{-1}W(v')dv'}. \tag{18}$$

Note that this expression is in fact independent of t. From (12a) and (12d), we can calculate the long term behavior of $f(x,v,t)$ in $D_+(t)$ and $D_-(t)$:

$$f(x,v,t)=N\frac{W(|v|)}{2|v|\int_0^\infty v'^{-1}W(v')dv'}=N\phi(v), \tag{19}$$

which is the steady state solution in (11).

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The result is that if one chooses a distribution $W(v)$ with $W(0)=0$ (i.e., with $\alpha>0$), the elastic gas can end up with a velocity distribution function proportional to $\phi(v)$. Notice that in the case $0<\alpha<1$, even though ϕ is singular at $v=0$ [as $\phi(v)\sim v^{\alpha-1}$], it is still integrable and therefore (11) is an acceptable solution.

A gas of elastic particles at a constant temperature T has uniformly distributed positions and a Gaussian distribution of velocities, so $f(x,v)\propto \exp(-v^2/2T)$. In order to mimic such behavior in our system, Equation (11) demonstrates that the energy source must then be proportional to $v\ \exp(-v^2/2T)$, in other words, $\alpha=1$.

2. Solution of the elastic model when $\alpha=0$

Reference 5 considers the case $W(v)\sim \exp(-v^2/2T)$, i.e., $\alpha=0$. Here we show that such a system never reaches a steady state, but that its velocity distribution does tend to resemble the non-normalizable density $W_0(|v|)/|v|$ as time increases.

If $W(0)\neq 0$ then the first term in the expansion of $\mathscr{D}(p)$ is

$$\mathscr{D}(p)\approx 2W(0)p\ln(1/p); \tag{20}$$

see Appendix A for the calculation. Therefore $\tilde{R}(p)\approx N/2pW(0)\ln(1/p)$, which gives, using well known results[17] on the inverse Laplace transform,

$$R(t)\approx \frac{N}{2W(0)\ln(t)}. \tag{21}$$

As $t\to\infty$, the solution in $D_+(t)\cup D_-(t)$, which never contains the line $v=0$, but does grow to cover the remainder of the phase space, is

$$f(x,v,t)\approx \frac{N}{2W(0)\ln(t)}\frac{W(|v|)}{|v|}. \tag{22}$$

The function $f(x,v,t)$ becomes increasingly peaked around $v=0$: the particles have a tendency to form a cluster at rest, even though there is no dissipation in the system. Because there is no steady state that satisfies the boundary conditions, the time dependence of the solution is essential: the medium is continually cooling even though there is no dissipation.

V. THE QUASI-ELASTIC LIMIT

We now consider a coefficient of restitution $r<1$ but such that $(1-r)N\ll 1$ in order to avoid inelastic collapse. This is the quasi-elastic limit described in Ref. 16. One expects the solution of this problem to be close to the solution of the elastic one. Therefore, in this section, we study the weakly inelastic case as a perturbation of the elastic gas solution in (11). As this steady state solution does not exist for $\alpha=0$, this section treats only $\alpha>0$. Let $g(x,v,t)$ be defined by

$$f(x,v,t)=N(\phi(v)+g(x,v,t)), \tag{23}$$

and assume that this perturbation g is very small compared to ϕ; in fact we expect it to be of order $qN=(1-r)N/2$. All calculations in this section are done to first order in qN.

The Boltzmann equation (4) from Section III becomes

$$g_t+vg_x+qNS(v)=0, \tag{24}$$

where

$$S(v)\equiv \frac{d}{dv}\left[\int_{-\infty}^{\infty}|v'-v|(v'-v)\phi(v)\phi(v')dv'\right] \tag{25}$$

is the first order contribution of the acceleration term. The boundary conditions (6) and (7a) are

$$g(1,v,t)=g(1,-v,t),\quad \forall v, \tag{26}$$

$$vg(0,v,t)=R_g(t)W(v),\quad \forall v>0, \tag{27}$$

where we have defined

$$R_g(t)\equiv \int_{-\infty}^{0}|v'|g(0,v',t)dv', \tag{28}$$

now to be the perturbation to the rate of collisions with the wall at $x=0$.

A. Slowly evolving behavior of the quasi-elastic medium

A steady state solution must satisfy

$$vg_x+qNS(v)=0,$$

and the boundary conditions (26) and (27). Neglecting the solution representing a cluster at rest, this is easily solved by $g(x,v)=-qNS(v)C(x,v)/|v|+G(v)$, where

$$C(x,v)\equiv\begin{cases}x, & v>0,\\ 2-x, & v<0.\end{cases}$$

The form of $G(v)$ is determined by the boundary conditions: (26) implies that $G(v)$ is even, while (27) gives $G(v)=R_gW(|v|)/|v|$. The normalization condition (8) implies that $\int_{-\infty}^{\infty}\int_0^1 g(x,v)dx\ dv=0$, so R_g must be given by

$$R_g=qN\frac{\int_0^\infty v'^{-1}S(v')dv'}{\int_0^\infty v'^{-1}W(v')dv'}. \tag{29}$$

Again, the integrability of this expression is what determines the existence of a steady state solution. Appendix B demonstrates that $S(v)\approx Bv^{\alpha-1}$ as $v\to 0$. Thus $S(0)\neq 0$ for all $\alpha\leqslant 1$ and R_g is not well-defined. If, however, $\alpha>1$, there does exist a steady state perturbed solution:

$$f(x,v)=N\left\{\phi(v)\left[1+2qN\int_0^\infty\frac{S(v')}{v'}dv'\right]-qN\frac{S(v)}{|v|}C(x,v)+\mathscr{O}(q^2N^2)\right\}. \tag{30}$$

Since $2qN\int_0^\infty v'^{-1}S(v')dv'$ is negative, its effect is to "remove particles" from the elastic distribution $N\phi$ and to redistribute them into the second term, a distribution proportional to $-S(v)/|v|$. This function is positive near zero and negative for larger values of v, so the number of particles with low speeds is increased. As expected, the system is "slower" (lower total energy) in this inelastic steady state than in the elastic case.

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This section has shown that the shape of the inelastic steady state solution depends on only two parameters: qN and α [through $\phi(v)$ and $S(v)$]. The calculations were done under the assumption that $g(x,v) \ll \phi(v)$, i.e., $qN << 1$. This inequality is satisfied, except at very small velocities. For $|v| \ll 1$ both $\phi(v)$ and $S(v)$ are proportional to $|v|^{\alpha-1}$. Therefore the second term on the right-hand side of (30) is $\mathcal{O}(qN)\mathcal{O}(|v|^{\alpha-2})$ while the first term is $\mathcal{O}(|v|^{\alpha-1})$. Thus if $|v|$ is of order qN, the two terms are of the same magnitude and the treatment in this section is no longer valid. We shall see in Section VII that the cure for this breakdown involves the creation of a slowly varying state with $|f_t| \ll |v f_x + (af)_v|$. In this limit, for short time intervals, this section's steady state distribution provides a fairly accurate description of the system.

B. Time dependent solution for the quasi-elastic medium

In this subsection, we discuss a time dependent solution of (24) that elucidates the case $0 < \alpha \leq 1$. The techniques employed mirror those used in Section IV B, so we will just state the results. The behavior of the system is best understood by looking at $R_g(t)$, the perturbation to the rate of collisions with the heated wall. For $\alpha > 1$ and $t \to \infty$, the time dependent calculation reproduces the steady state result in (29). However, for $0 < \alpha < 1$, we find that, for large t,

$$R_g(t) \approx \frac{2^\alpha qNBt^{1-\alpha}}{\alpha(1-\alpha^2)\int_0^\infty v'^{-1}W(v')dv'}$$

$$= \frac{-2^\alpha qNA}{(1-\alpha^2)}\left(\int_0^\infty \frac{W(v')}{v'}dv'\right)^{-3} t^{1-\alpha}. \qquad (31)$$

Therefore $R_g(t)$ is growing with time like $t^{1-\alpha}$, which eventually contradicts the assumption of a small perturbation. Thus, when $0 < \alpha < 1$, although there exists a steady state of the *elastic* gas, the slightest dissipation destroys it. The mathematical reason for this breakdown is that the function $S(v)/v$ is not integrable. Because $W(v)$ is not ''flat enough'' at $v=0$, the wall produces too many particles with small speeds to sustain a steady state.

The case $\alpha=1$ is special for two reasons: (1) $\alpha=1$ is a critical value above which there exists a steady state of the inelastic medium, and (2) $\alpha=1$ produces an elastic gas with a Maxwellian velocity distribution. Here, we find that, for large t,

$$R_g(t) \approx \frac{qNB\ln(t)}{\int_0^\infty v'^{-1}W(v')dv'}$$

$$= -qNA\left(\int_0^\infty \frac{W(v')}{v'}dv'\right)^{-3} \ln(t). \qquad (32)$$

As in the previous case, $R_g(t)$ grows over time, the assumption that $g(x,v,t)$ is a small perturbation breaks down, and there is no steady solution for the dissipative system.

Note that, since $\lim_{\alpha\to 1^-}(t^{1-\alpha})/(1-\alpha) = \ln$, the $R_g(t)$ derived for the $\alpha=1$ case (32) can in some sense be perceived as the limit of the $R_g(t)$ in (31) calculated for $0 < \alpha < 1$.

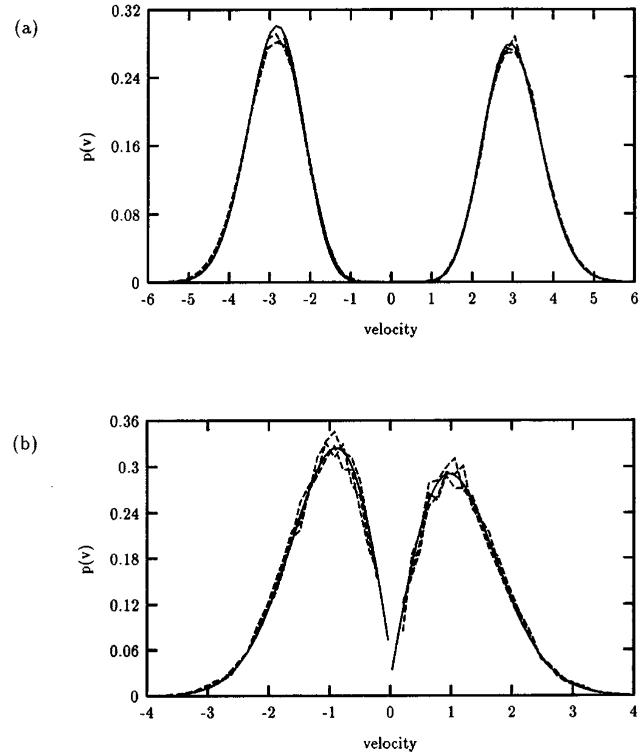

FIG. 3. These figures show the agreement between simulations and the analytic descriptions of the metastable states. The solid line is theory and the dashed lines are the velocity distributions for three successive time intervals in the simulations. In both cases, the system has $N=100$ and is allowed to run for a total of twenty million collisions. The velocity sampling is done during the last six million of these collisions. (a) No clumping: Here $\alpha=10$ and $qN=0.02$. (b) Clumping: Here $N_c/N=0.53$, $\alpha=2$, and $qN=0.03$. There are no simulational values at small velocities, as any data on slowing moving particles that are part of N_f is masked by the relatively stationary clump.

VI. THE BOLTZMANN EQUATION REVISITED

The analytic solution derived above corresponds to a uniform spatial distribution of the particles and a velocity distribution that is a slightly skewed version of $\phi(v)$. While this sort of state has been observed computationally for very high α and small qN [see Fig. 3(a)], most simulations show more complicated long term behavior. Note that throughout this paper, the simulations are done with an event-driven code and the initial conditions are always $f(x,v)=N\phi(v)$: uniform spatial distribution between zero and one, and a velocity distribution that matches that of the elastic solution.

In the analytic steady state solution in (30), all N particles in the system are in motion. The velocity distribution is approximately $\phi(v) \propto |v|^{\alpha-1}\exp\{-v^2/2\}$, so the probability of having a low speed is small. However, simulations have shown, in addition to the states that match (30), other states in which N_c of the particles form a stationary clump against the elastic wall at $x=1$. In other words, N_c particles have positions roughly equal to one and velocities approximately equal to zero, while the remaining $N_f \equiv N - N_c$ ''free'' particles move relatively rapidly and are distributed uniformly through the system. One can no longer expect the Boltzmann equation to describe the behavior of the N_c particles in the clump, as correlations are known to develop in high density

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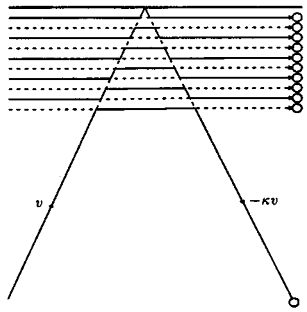

FIG. 4. The collision wave model for a fast particle's interaction with the clump and the elastic wall. In this picture, $N_c=10$ and $q=0.005$. The horizontal axis is time, the vertical axis is position, and the heavy line at the top is the elastic wall. The alternating solid and dashed lines are the world lines of the particles. A fast particle enters the clump from below with velocity v and is ejected with velocity $-\kappa v$.

regions.[4] However, for the N_f free particles, the Boltzmann treatment of the previous section should still hold, albeit with slightly different boundary conditions at $x=1$. The clump at $x=1$ acts as a cushion against the elastic wall so that when a free particle with velocity v strikes the N_c particles next to the wall, a series of inelastic collisions occurs within the clump. Energy is dissipated, and the particle is ejected with a velocity $v'=-\kappa v$, where the effective coefficient of restitution, κ, depends on the size of the clump (N_c) and the degree of inelasticity (q):

$$\kappa=\frac{2(1-q)^{2N_c}-q}{2-q}\approx 1-2qN_c. \tag{33}$$

The above expression is calculated by assuming that the particles in the clump are basically stationary compared to the incoming particle and hence the collisions occur in a wave[6,7] (see Fig. 4). Now the clump can be replaced by an *inelastic* wall at $x=1$, and we want to solve the time independent Boltzmann equation with the old boundary condition (7a) at $x=0$, but with a new boundary condition at $x=1$:

$$f(1,v)=\kappa^2 f(1,-\kappa v),\quad \text{for all } v>0. \tag{34}$$

The above techniques can be used to solve for a steady state $f(x,v)$ to first order in q:

$$\begin{aligned} v>0&: f(x,v)=N_f\left[\beta\phi(v)-qN_f x\frac{S(v)}{v}\right],\\ v<0&: f(x,v)=N_f\left[\frac{\beta}{\kappa^2}\phi(v/\kappa)-\frac{qN_f}{|v|}\right.\\ &\qquad\left.\times\left((1-x)S(v)+\frac{1}{\kappa}S\left(\frac{v}{\kappa}\right)\right)\right], \end{aligned} \tag{35}$$

where

$$\beta\equiv 1+2qN_f\int_0^\infty \frac{S(v')}{v'}dv'-qN_c \tag{36}$$

is a constant. Now we can look at the distributions we see in the simulations where clumps appear and match them to the above analytic form [see Fig. 3(b)]. Note that when $N_c=0$ (no clump) or $q=0$ (totally elastic), then $\kappa=1$, and $f(x,v)$ reduces to the previous solution (30). Again all calculations have been done to first order in qN_f and qN_c. Also, the breakdown that occurs at small velocities $[v=\mathcal{O}(qN_f)]$ for the clumpless steady state solution described in Section V A is still an issue here.

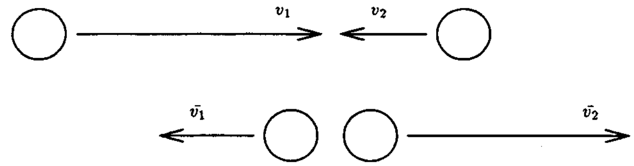

FIG. 5. A collision. This interaction can be viewed as being between two particles (in which case $v_1\to\overline{v_1}$ and $v_2\to\overline{v_2}$), or as being between two velocities (in which case $v_1\to\overline{v_2}$ and $v_2\to\overline{v_1}$).

VII. CLUMP FORMATION

As the simulations shown in Fig. 3 demonstrate, on intermediate time scales, the solution derived above provides a relatively good description of the system. However, when observed over long periods of time, it becomes clear that the clump is growing. As N_c increases, the system moves among the metastable states that are described by the distributions (35) with different values of N_c. Thus to understand the time dependence of the system, we must understand the mechanics of clump formation and growth.

When two elastic particles collide, they exchange velocities. Thus, instead of a group of particles rattling back and forth, the system can be viewed as a collection of interacting velocities. When $q=0$, the collisions are elastic, and hence do not affect the velocities at all. Each velocity propagates independently through the system. In the quasi-elastic regime ($q\ll 1$), this picture is still approximately valid. However, at each collision, the velocities are slightly altered. Consider two particles approaching each other with velocities v_1 and v_2 (see Fig. 5). If $q=0$, after the collision, the velocities are $\overline{v_1}=v_2$ and $\overline{v_2}=v_1$, and it is as if the particles have passed through each other: the collision might as well never have occurred. If however q is nearly, but not quite, zero, then $\overline{v_1}=pv_2+qv_1$ and $\overline{v_2}=pv_1+qv_2$. Instead of attaching each velocity to a particle, view this as merely a rightward moving velocity and a leftward moving velocity, and calculate the change in each due to the collision:

$$\begin{aligned} &\text{right-moving:}\quad \overline{v_2}-v_1=pv_1+qv_2-v_1=q(v_2-v_1),\\ &\text{left-moving:}\quad \overline{v_1}-v_2=pv_2+qv_1-v_2=q(v_1-v_2). \end{aligned} \tag{37}$$

Thus q measures the strength of the interaction between velocities.

Every time a particle hits the heated wall, a new velocity ($v_{\rm new}>0$) is added to the system and an old velocity ($v_{\rm old}<0$) is retired. The ''aging'' trajectory of a typical velocity can be viewed as follows: since $v_{\rm new}>0$, the velocity will move away from $x=0$ toward the elastic wall. As it goes, it will lose energy due to interactions with other velocities, i.e., collisions, and slow down. At the far wall, it will be reflected, thus becoming negative, and begin its jour-

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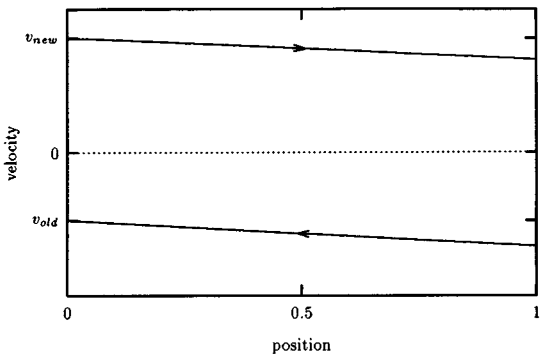

FIG. 6. The aging process in phase space for a velocity that is born at the heated wall ($x=0$) with velocity $v_{\rm new}$. It loses energy as it moves through the system (top line) toward the elastic wall ($x=1$), where it is reflected. Still losing energy, it then returns (bottom line) to the heated wall, where its final velocity, $v_{\rm old}$, is retired.

ney back toward the heated wall, still losing energy along the way. It gets back to $x=0$ with $v_{\rm old}<0$, where $|v_{\rm old}|<v_{\rm new}$. In Fig. 6, a typical aging trajectory is traced out in (x,v) phase space. If one examines the interaction between the velocities (37) closely, one sees that in fact a velocity can gain, as well as lose, due to a collision. However, in a system where most of the particles are moving at approximately the same speed, any given velocity is much more likely to collide with a velocity moving in the other direction. Such an interaction pulls both velocities closer to zero. Note that this description of the aging trajectory makes no assumptions about the boundary conditions at $x=1$; all that is needed is that the velocity changes sign, which occurs no matter the size of the clump at that wall. From this description, we can better understand the skewing we see in the distribution when $q\neq 0$. We now know that the leftward-moving particles ($v<0$) are going, on the whole, slightly slower than their rightward-moving counterparts. This is consistent with the fact that the velocity distribution calculated from $f(x,v)$ has a peak for negative velocities that is slightly closer to zero than the peak for positive velocities.

Most velocities born at the wall $x=0$ follow trajectories like that described above and hence produce the solution $f(x,v)$. In fact, the Boltzmann treatment implicitly assumes that all velocities do, within some finite time, come in contact with the heated wall and hence are replaced. However, this picture breaks down upon the emittance by the wall of a very slow particle. The aging process described above is one of a continual decrease in speed. Thus the velocity may go to zero *before* it can return to the heated wall. Once this happens, the velocity is carried on a tide back to the elastic wall at $x=1$ where it remains—a newly recruited member of the clump. This phenomenon explains the breakdown of the analytic form of $f(x,v)$ at small velocities that was described at the end of Section V A. Note that the argument given above as to why the interactions between velocities slow everyone down is not valid for a particle that is moving much more slowly than the average speed. In this case, the number of particles going each way that the slow velocity, call it v_s, encounters is determined by the distribution $f(x,v)$. If both the velocity and number density distributions are symmetric at every point in space, then the gains and losses due to collisions average out to zero. However, in our system, when $q\neq 0$, $f(x,v)$ is asymmetric in both position and velocity space, and thus a slow velocity undergoes a net drift through the system. This phenomenon can be quantified by calculating $a(x,v_s)$, the acceleration of a particle at (x,v_s) in the phase space. From (5) we have

$$a(x,v_s)\equiv q\int_{-\infty}^{\infty}|v'-v_s|(v'-v_s)f(x,v')dv'$$
$$=2q\left\{-2v_s\int_0^{\infty}v'f_S(x,v')dv'+\int_0^{\infty}(v'^2+v_s^2)\right.$$
$$\left.\times f_A(x,v')dv'-\int_0^{v_s}(v'-v_s)^2f(x,v')dv'\right\},$$

where $f_S(x,v')\equiv\frac{1}{2}[f(x,v')+f(x,-v')]$ and $f_A(x,v')\equiv\frac{1}{2}[f(x,v')-f(x,-v')]$ are the symmetric and anti-symmetric parts of the distribution. We are interested in the regime in which the analytic solution breaks down, so we know that v_s is $\mathscr{O}(qN_f)$. However, the particles encountered are moving rapidly (v'), so it is valid to use the analytic description (35) for $f(x,v')$. The above expression for the acceleration can then be simplified by keeping only terms of order $v_s(qN_f)$ or $(qN_f)^2$. Now we are in a position to calculate a differential equation for the slow velocity v_s:

$$\frac{dv_s}{dx}=\frac{dt}{dx}\frac{dv_s}{dt}=\frac{1}{v_s}a(x,v_s)$$
$$\approx(qN_f)\left\{-4\beta\int_0^{\infty}v'\phi(v')dv'+\left(\frac{qN_f}{v_s}\right)\right.$$
$$\left.\times\left[\beta\alpha\frac{qN_c}{qN_f}+2(1-x)\int_0^{\infty}v'S(v')dv'\right]\right\}, \qquad (38)$$

where β is as defined in (36). This equation describes the evolution of a slow velocity through the system where the other velocities are assumed to be drawn from the distribution $f(x,v)$.

The key to clump formation is the critical velocity, $v_{\rm crit}$. Any velocity that is emitted from the wall which is greater than $v_{\rm crit}$ will make it back to the heated wall, any lesser velocity will not, and hence will become part of the clump. Specifically, given a v_s^o at $x=0$, use (38) to calculate $v_s(x=1)$. Reflection through any pre-existing clump and off the elastic wall will make it $\overline{v_s}(x=1)=-\kappa v_s(x=1)$. Then use (38) again to integrate back to $x=0$. If, upon return, v_s is exactly zero, then v_s^o is $v_{\rm crit}$. Note that this value will be a function of the degree of inelasticity (q), the total number of particles in the system (N), the strength of the forcing (α), and the number of particles that remain free (N_f). However, only three parameters appear explicitly in the formulation: α, qN_f, and $qN_c=qN-qN_f$.

It was mentioned in the Introduction that the size of the clump and its average energy are proportional to $(1-r)$. This implies that for a completely elastic system, there should be a totally stationary, infinitely narrow bunch of particles up against the elastic wall. However, for clump formation to occur, particles with $v<v_{\rm crit}$ must be produced by the

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heated wall. Since $dv_s/dx \propto q=(1-r)/2$, when $r=1$, $dv_s/dx=0$, and $v_{\rm crit}$ is zero. Therefore no clump can form in the totally elastic case; $r\to 1$ is a singular limit.

VIII. TIME EVOLUTION

At intermediate times, the system is well described by the distribution (35), where the exact form of $f(x,v)$ depends on α, qN_f, and qN_c. However, while q,α, and $N=N_f+N_c$ are fixed parameters of the system, the clump size, $N_c(t)$, evolves in time. To truly describe the long term behavior of the system, there must be a prediction for the variation of $N_c(t)$, or equivalently of $N_f(t)=N-N_c(t)$, as a function of time.

At any given time, t, there are $N_f(t)$ free particles, and so we can calculate $v_{\rm crit}(\alpha, qN_f, qN_c=qN-qN_f)$. Since the distribution function $W(v)$ for emission from the wall is known, the probability that a velocity less that $v_{\rm crit}$ will be produced can be written:

$$p(v<v_{\rm crit})=\int_0^{v_{\rm crit}} W(v')dv'. \tag{39}$$

In a small time interval Δt, the number of particles added to the clump (and hence removed from N_f) is just the number of velocities with $v<v_{\rm crit}$ produced in Δt, i.e., the number of times the heated wall is hit (C_w) multiplied by $p(v<v_{\rm crit})$. Assuming that dC_w/dt is known, the change in N_f is given by

$$\Delta N_f \approx -\left(\frac{dC_w}{dt}\right)\Delta t p(v<v_{\rm crit}),$$

so that, in the limit as $\Delta t\to 0$,

$$\frac{dN_f}{dt}\approx -\frac{dC_w}{dt}\int_0^{v_{\rm crit}} W(v')dv', \tag{40}$$

where the dependence on qN_f is hidden in $v_{\rm crit}$ and dC_w/dt.

In fact, dC_w/dt is merely the rate at which particles hit the left-hand wall: this is just the R that is used to define the boundary conditions in (7b). Therefore

$$\begin{aligned}\frac{dC_w}{dt}=R&=\int_{-\infty}^0 |v'|f(0,v')dv'\\ &=N_f\left(1+2qN_f\int_0^\infty \frac{S(v')}{v'}dv'-qN_c\right)\\ &\quad\times\int_0^\infty v'\phi(v')dv',\end{aligned} \tag{41}$$

where the time dependence is hidden in N_f and N_c.

We now have all the ingredients necessary to calculate $d(qN_f)/dt$, given α, qN_f, and $qN_c=qN-qN_f$. The evolution of the system is described by

$$\frac{d(qN_f)}{dt}=F_{\{qN,\alpha\}}(qN_f), \tag{42}$$

where $F_{\{qN,\alpha\}}$ is a function depending only on the forcing W_α and qN. Thus those two parameters, plus the initial condition, determine the time evolution of the system (i.e., systems with different N and q will undergo the same evolution due to the forcing W_α as long as they have the same qN). Note that $v_{\rm crit}$ must be determined numerically, so $d(qN_f)/dt$ is a complicated function of qN_f and the other variables. However, the initial conditions are such that the system starts clumpless, i.e., at $t=0$, $N_f=N$, and we can integrate numerically to find $qN_f(t)$. Figure 7 shows a typical result; the dots are points from six simulations, the solid line is the prediction of the above analytic formulation. Note that the initial distribution is always $f(x,v)=N\phi(v)$; the variation in the runs is due to changes in the exact initial configuration as well as in the random seed for generation of $W(v)$ at the heated wall. We chose these initial conditions because they were likely to produce a slowly growing clump in contrast to an initial state that already includes many slowly moving particles. We are confident that the algebra developed here could be used to describe the time evolution starting from a wide range of initial conditions.

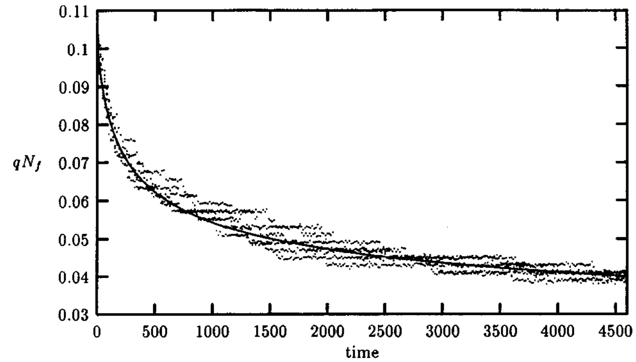

FIG. 7. A comparison of simulational and theoretical values for $qN_f(t)$. The solid line is obtained by numerical integration of (40). The data is from six different runs of fifteen million collisions each. As always, the initial distribution was $f(x,v)=N\phi(v)$. Here $\alpha=6$, $N=50$, and $q=0.002$.

IX. CONCLUSION

This work has examined the density variations in a one-dimensional system of N inelastic particles. These particles are constrained to move on the line between an elastic wall at $x=1$ and an energy source at $x=0$. This system can be described by a Boltzmann equation (4). Analytic considerations demonstrate that this equation has a true steady state solution in only one case: when $q=0$ and $\alpha>0$. This is a system of perfectly elastic particles with $W(0)=0$, i.e., the boundary forcing forbids the return of particles with no kinetic energy. In all other cases, there is no completely stable steady state solution.

In this paper we have also described a set of approximate solutions for the $q\neq 0$ case. These distributions are *almost* steady state, and are derived for more general boundary conditions than are used in the basic Boltzmann treatment. This new approach allows us to match the analytics to numerical simulations that show the system developing a clump of $N_c(t)$ nearly stationary particles against the elastic wall. The remaining $N_f(t)=N-N_c(t)$ free particles are distributed uniformly through the system. There is a mechanism for

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clump growth, but not one for clump shrinkage. Thus, even for a system that begins with $N_f = N$ (no clump), we see that N_f approaches 0 as $t \to \infty$ and the particle density at all points other than $x=1$ goes to zero. In fact, there is no way for the last rapidly moving particle to be absorbed into the clump. Therefore, the final state of the system has $N_c = N-1$, and a lone free particle oscillating between the clump and the heat source. This is the state observed in Ref. 5. This work develops an analytic description of the mechanism of clump formation that allows us to predict the shrinkage of N_f over time. Thus we claim to understand the development of density and velocity variations in a one-dimensional system of inelastic particles.

The analytic work in this paper has focused on a model for the energy source at $x=0$ in which a particle that collides with this wall is returned to the system with a velocity from a fixed distribution. Thus the velocity of a particle ejected from the wall is independent of the velocity with which it approached the wall. This boundary condition is an idealized cousin of the oscillating wall used in work done on similar one-dimensional systems. However, the sort of clumping behavior we observe has also been seen for particles interacting with a shaking wall.[5] In works where a gravitational field is included,[8,10] gravity serves to drive the particles back toward the energy source, as the elastic wall at $x=1$ does in our system, and again clumping is observed. Therefore, the behavior studied in this paper and the description of the mechanisms at work seem to be general characteristics of driven inelastic granular systems.

ACKNOWLEDGMENTS

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APPENDIX A: APPROXIMATION OF $\mathcal{D}(P)$ FOR SMALL p AND $\alpha=0$

In this appendix we calculate an approximation for

$$\mathcal{D}(p) = 1 - \int_0^\infty W(v')\exp\left(\frac{-2p}{v'}\right)dv', \tag{A1}$$

for $p \ll 1$ when $\alpha=0$. Note that $\mathcal{D}(0)=0$ and the derivative

$$\mathcal{D}'(p) = 2\int_0^\infty v'^{-1} W(v')\exp\left(\frac{-2p}{v'}\right)dv'$$

is not defined at $p=0$ as $W(0) \neq 0$. Thus $\mathcal{D}(p)$ does not have a Taylor expansion about $p=0$. Now choose a δ such that $p \ll \delta \ll 1$ and split the integral into two parts:

$$\begin{aligned} I_1(p,\delta) &\equiv \int_0^\delta W(v')\exp\left(\frac{-2p}{v'}\right)dv', \\ I_2(p,\delta) &\equiv \int_\delta^\infty W(v')\exp\left(\frac{-2p}{v'}\right)dv', \end{aligned} \tag{A2}$$

so $\mathcal{D}(p) = 1 - I_1(p,\delta) - I_2(p,\delta)$.

In the first integral, we use the fact that $W(v') \approx W(0)$. A change of variables, $u = 2p/v'$ gives

$$\begin{aligned} I_1(p,\delta) &\approx 2pW(0)\int_{2p/\delta}^\infty u^{-2}\exp(-u)\,du \\ &\approx 2pW(0)\left[\frac{1}{(2p/\delta)} + \ln\left(\frac{2p}{\delta}\right)\right], \end{aligned}$$

where the second approximation is from the power series for the incomplete gamma function.[18]

To tackle the second integral, we expand the exponential in powers of p/v since $p/v < p/\delta \ll 1$:

$$\begin{aligned} I_2(p,\delta) &\approx \int_\delta^\infty W(v')\left(1 - \frac{2p}{v'}\right)dv' \\ &= \int_0^\infty W(v')dv' - \int_0^\delta W(v')dv' \\ &\quad - 2p\int_\delta^\infty \frac{W(v')}{v'}dv' \\ &\approx 1 - W(0)\delta - pW(0)\int_{\delta^2/2}^\infty u^{-1}\exp(-u)\,du \\ &\approx 1 - W(0)\delta - pW(0)\left[-\ln\left(\frac{\delta^2}{2}\right)\right], \end{aligned}$$

where the last approximation is again due to the expansion of the incomplete gamma function. Now, as long as $I_1 + I_2$ is independent of δ, we can combine these results to find

$$\begin{aligned} \mathcal{D}(p) &\approx 1 - [W(0)\delta + 2W(0)p\ln(2p/\delta)] - [1 - W(0)\delta \\ &\quad + pW(0)\ln(\delta^2/2)] \approx -2W(0)p\ln(p). \end{aligned} \tag{A3}$$

This matching technique is documented in Ref. 18.

APPENDIX B: ESTIMATE OF $S(v)$ FOR SMALL VALUES OF v

In this appendix we calculate the behavior for small v of

$$S(v) = \frac{d}{dv}(a\phi), \tag{B1}$$

where

$$a(v) = \int_{-\infty}^\infty |v'-v|(v'-v)\phi(v')dv'. \tag{B2}$$

Rewrite this expression by splitting the integral to get rid of the absolute value sign. Then take advantage of the evenness of $\phi(v)$ to group the integration into large v' and small v' ranges:

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$$a(v)=\int_v^\infty (v'-v)^2\phi(v')dv'-\int_{-v}^{v}(v'-v)^2\phi(v')dv'$$
$$-\int_{-\infty}^{-v}(v'-v)^2\phi(v')dv'$$
$$=-4v\int_v^\infty v'\phi(v')dv'-\int_{-v}^{v}(v'-v)^2\phi(v')dv'.$$

Therefore, when $v\to 0$, $a(v)\sim -4v\int_0^\infty v'\phi(v')dv'$. Since

$$\phi(v)=\frac{W(|v|)}{2|v|\int_0^\infty v'^{-1}W(v')dv'},$$

and $W(|v|)\sim Av^\alpha$ for $v\ll 1$, we have

$$a(v)\phi(v)$$
$$\sim -4v\frac{\int_0^\infty W(v')dv'}{2\int_0^\infty v'^{-1}W(v')dv'}\frac{Av^\alpha}{2|v|\int_0^\infty v'^{-1}W(v')dv'}$$
$$=-Av^\alpha\left(\int_0^\infty v'^{-1}W(v')dv'\right)^{-2}, \tag{B3}$$

which implies that $S(v)\sim Bv^{\alpha-1}$ for $v\ll 1$, where B is defined by

$$B=-\alpha A\left(\int_0^\infty \frac{W(v')}{v'}dv'\right)^{-2}, \tag{B4}$$

where A is the normalization constant for $W(v)$, i.e., $W(v)\approx Av^\alpha$ as $v\to 0$.

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