ULTRASOUND AS A PROBE OF PLASTICITY? THE INTERACTION OF ELASTIC WAVES WITH DISLOCATIONS

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An overview of recent work on the interaction of elastic waves with dislocations is given. The perspective is provided by the wish to develop non-intrusive tools to probe plastic behavior in materials. For simplicity, ideas and methods are first worked out in two dimensions, and results in three dimensions are then described. These results explain a number of recent, hitherto unexplained, experimental findings. The latter include the frequency dependence of ultrasound attenuation in copper, the visualization of the scattering of surface elastic waves by isolated dislocations in $LiNbO_3$, and the ratio of longitudinal to transverse wave attenuation in a number of materials.

Specific results reviewed include the scattering amplitude for the scattering of an elastic wave by a screw, as well as an edge, dislocation in two dimensions, the scattering amplitudes for an elastic wave by a pinned dislocation segment in an infinite elastic medium, and the wave scattering by a sub surface dislocation in a semi infinite medium. Also, using a multiple scattering formalism, expressions are given for the attenuation coefficient and the effective speed for coherent wave propagation in the cases of anti-plane waves propagating in a medium filled with many, randomly placed screw dislocations; in-plane waves in a medium similarly filled with randomly placed edge dislocations with randomly oriented Burgers vectors; elastic waves in a three dimensional medium filled with randomly placed and oriented dislocation line segments, also with randomly oriented Burgers vectors; and elastic waves in a model three dimensional polycrystal, with only low angle grain boundaries modeled as arrays of dislocation line segments.

Keywords: Dislocations, Elastic waves, Plasticity.

I. INTRODUCTION

The interaction of acoustic waves with dislocations was the subject of many detailed studies in the period from the early 1950's to the mid 1980's, with the theory of Granato and Lücke [1956 a;b] (hereafter GL) widely held as the standard theory to this day. To be sure, the GL theory has had many successes that have withstood the test of time [Granato & Lücke, 1966]: It describes particularly well damping, internal friction and modulus change in solids. However, it is less successful in the explanation of, say, thermal conductivity measurements [Anderson, 1983].

From a theoretical point of view, the GL theory builds on the string model of Koehler [1952] in which a pinned dislocation segment is treated as a scalar string that vibrates around an equilibrium, straight, position. Also, it is a mean-field theory that looks for an effective velocity and attenuation coefficient for a coherent scalar wave that propagates in a medium filled with many such dislocations. Since it is a scalar theory, it cannot treat simultaneously the three polarizations of an elastic wave, of which the acoustic wave is but one, the longitudinal. Also, being a mean field theory, it cannot describe the interaction of an acoustic wave with a single, isolated dislocation. Recently, however, experiments have been able to distinguish quite clearly the attenuation of longitudinal waves from that of transverse waves, using Resonant Ultrasound Spectroscopy—RUS [Ledbetter & Fortunko, 1995; Ogi *et al.,* 1999; 2004; Ohtani *et al.,* 2005]. Also, stroboscopic X-ray topography experiments have been able to visualize quite clearly the scattering of 500 MHz surface elastic waves on $LiNbO_3$ off single dislocations [Shilo & Zolotoyabbko, 2002; 2003; Zolotoyabko *et al.,* 2001]. How is one to make quantitative sense of these experiments? GL theory needs to be extended.

From a different point of view, plasticity in crystals, particularly in metals, is widely understood to depend on dislocation behavior, and there are important problems in materials science, such as the brittle to ductile transition [Roberts, 2002], and fatigue [Suresh, 1998; Kenderian *et al.,* 2003a; 2003b], where there is wide agreement that dislocations play a significant, possibly determinant, role. From an engineering design point of view the issue is well controlled in the sense that structures can be, and are, successfully constructed. From a basic physics point of view, however, those phenomena are far from being understood, in the sense that current theoretical modeling has no predictive power: If a new form of steel, say, is fabricated, current theory cannot, to the best of our knowledge, make quantitative predictions concerning its mechanical properties as a function of temperature or cyclic loading. It is our opinion that one important cause for this lack of basic knowledge is the paucity of experimental measurements (as opposed to visualizations) concerning dislocations. This is so because dislocations need to be seen through transmission electron microscopy of samples that must be specially prepared [Xu *et al.,* 2005; Arakawa *et al.*, 2006; Robertson *et al.*, 2008]. It stands thus to reason that it would be desirable to have quantitative measurements carried out with non invasive probes. Is this possible? We believe acoustic, or ultrasonic, measurements may provide such quantitative data. In the pursuit of this research program, we found that the comparatively simple and basic theory problems of the behavior of elastic waves traveling through an elastic medium full of dislocations, or the interaction of an elastic wave with a single dislocations, were not well understood.

This paper provides a brief overview of our work in the past few years on the interaction of sound with dislocations.

II. THEORETICAL FRAMEWORK

We consider an homogeneous, isotropic, continuous elastic medium of density ρ and elastic constants $c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. The experimental results we have in mind for the theoretical framework to explain are performed in three dimensions, so indices will take three values: $i, j, k, \cdots = 1, 2, 3$. It is instructive however, as will be discussed below, to develop a theory in the simpler case of two dimensional solids, in which case indices will take the values $a, b, \cdots = 1, 2$. The state of such a system is described by a vector field $\vec{u}(\vec{x}, t)$, of (small, so that equations will be linear) displacements $\vec{u}$, at time t, of points whose equilibrium position is $\vec{x}$. Dealing with a continuum means that we shall consider length scales, particularly wavelengths, that are long compared with interatomic spacing. We shall consider both infinite and semi-infinite media, where wave propagation is governed by the usual wave equation

$$\rho\frac{\partial^2 u_i}{\partial t^2} - c_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} = 0 \quad (1)$$

with appropriate boundary conditions.

The elastic medium jus described supports the propagation of longitudinal (acoustic) and transverse waves that propagate with velocities $c_L = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_T = \sqrt{\mu/\rho}$, respectively. We shall call γ their ratio: $\gamma = c_L/c_T$. It is always greater than one.

Dislocations are modeled as one dimensional objects ("strings") $\vec{X}(s, t)$, where s is a Lagrangean parameter that labels points along the line, and t is time. They are characterized by a Burgers vector $\vec{b}$. In the present work, in three dimensions, we shall consider only dislocations whose motion consists of slight deviations from an equilibrium position that is a straight line, of length L, joining two fixed ("pinning") points, i.e., pinned dislocation segments. When $\vec{b}$ is parallel to the straight line, we have a screw dislocation; when it is perpendicular, an edge dislocation. They are endowed with mass per unit length m, and line tension Γ. Their (unforced) motion will be described by a conventional vibrating string equation

$$m\frac{\partial^2 X_i}{\partial t^2} + B\frac{\partial X_i}{\partial t} - \Gamma\frac{\partial^2 X_i}{\partial s^2} = 0 \quad (2)$$

with an *ad-hoc* friction term characterized by a first order time derivative with a phenomenological coefficient B. This term takes into account internal losses such as dislocation interaction with phonons and electrons. The coefficient B cannot be determined within the framework of continuum elasticity, and is a parameter to be adjusted by the data. Expressions for the mass per unit length m and line tension Γ can however be found [Lund, 1988]. For an edge dislocation they are

$$\begin{aligned} m &= \frac{\rho b^2}{4\pi}\left(1 + \gamma^{-4}\right)\ln\left(\frac{\delta}{\delta_0}\right) \\ \Gamma &= \frac{\rho b^2 c_T^2}{4\pi}\left(1 - \gamma^{-2}\right)\ln\left(\frac{\delta}{\delta_0}\right) \end{aligned} \quad (3)$$

where δ and δ_0 are external and internal cut-offs. The internal cut-off is roughly related to the distance from the dislocation core at which linear continuum elasticity ceases to be valid and the external cut-off is the distance to the nearest defect, or the sample size if there are no other defects. There are a number of assumptions needed to get these expressions, they have been spelled out in detail in Lund [1988], and will not be violated in the present work (see however the discussion in Section IV D) Note that the speed of propagation of perturbations along the string $\sqrt{\Gamma/m}$ (when the Lagrangean parameter s has dimensions of length) does not differ very much from the speed of propagation of elastic perturbations in the bulk.

Why should a dislocation scatter sound (more generally, elastic waves)? This question was addressed by Nabarro [1951] who identified two basic physical mechanisms: One, the stress associated with the wave would load the dislocation, to which the latter would respond by moving, which would in turn generate outgoing radiation. This mechanism is sometimes addressed in the literature as "secondary radiation generated by dislocation flutter". The second mechanism is through the nonlinear interaction generated by the probing of the wave of the region inside the dislocation core, where Eqns. (1) and (2) no longer hold. We expect this mechanism to be negligible when wavelengths are long compared to core radius, an assumption we have already made.

The motion of the dislocation under external loading is described by Eqn. (2) with a right hand side [Lund, 1988]:

$$m\frac{\partial^2 X_i}{\partial t^2} + B\frac{\partial X_i}{\partial t} - \Gamma\frac{\partial^2 X_i}{\partial s^2} = F_k \quad (4)$$

where

$$F_k(\vec{x},t) = \epsilon_{kjm}\tau_m b_i \sigma_{ij}(\vec{x},t)$$

is the usual Peach-Koehler [1950] force. All velocities are supposed to be small by comparison with the speed of wave propagation, so it is not necessay to worry about Lorentz-like forces. ϵ_{kjm} is the completely antisymmetric tensor of rank three (in two dimensions we shall use ϵ_{ab}, the antisymmetric tensor of rank two), τ_m is a unit tangent along the (moving!) dislocation line, and $\sigma_{ij}(\vec{x},t)$ is the space- and time-dependent external stress which, in Eqn. (4) is supposed to be evaluated at the dislocation position, $\vec{x} = \vec{X}(s,t)$. As has been noted, in three dimensions we shall consider pinned dislocation segments, so the boundary conditions needed to solve (4) are fixed endpoints. In two dimensions the dislocation is a point, the equation is an ordinary differential equation and no boundary conditions are needed. The solution is found using standard techniques [Maurel *et al.*, 2005a]: go to the frequency domain and write the solution as a superposition of the normal mode of the string.

Once the dislocation moves, it will generate waves. How will it do it? By way of Eqn. (1) suitably modified by a right hand side source term. It was found by Mura [1963] that this behavior was best described not in terms of particle displacement $\vec{u}$ but in terms of particle velocity $\vec{v} = \partial\vec{u}/\partial t$:

$$\rho\frac{\partial^2 v_i}{\partial t^2} - c_{ijkl}\frac{\partial^2 v_k}{\partial x_j \partial x_l} = s_i \quad (5)$$

where the source term s_i is, when a single dislocation is present,

$$s_i(\vec{x},t) = c_{ijkl}\epsilon_{mnk}\int_{\mathcal{L}} ds\, \dot{X}_m(s,t)\tau_n b_l \frac{\partial}{\partial x_j}\delta(\vec{x}-\vec{X}(s,t)).$$

When many dislocations are present, it is enough to take a sum over all of them.

In the deterministic case, i.e. when the right hand side of (5) is precisely known, its solution is given by a convolution of the Green's function G^0_{im} (also called impulse response) of the medium with the source s_i:

$$v_i(\vec{x},t) = \int d^3x' \int dt' G^0_{im}(\vec{x}-\vec{x}', t-t')s_m(\vec{x}',t'), \quad (6)$$

the Green function G^0_{im} being the solution to the equation

$$\rho\frac{\partial^2}{\partial t^2}G^0_{im}(\vec{x},t) - c_{ijkl}\frac{\partial^2}{\partial x_j \partial x_l}G^0_{km} = \delta_{im}\delta(\vec{x})\delta(t) \quad (7)$$

with appropriate boundary conditions. We shall consider infinite two dimensional space, infinite three dimensional space, and semi-infinite three dimensional space with a stress-free boundary.

When very many dislocations are present, it is not practical to consider a deterministic approach, a statistical one been more profitable and it will be discussed in Section IV.

III. SCATTERING BY A SINGLE DISLOCATION

As it was mentioned above, the ideas and methods are best explained in a step-by-step way, starting in two dimensions. There is one caveat in this case, namely that wave propagation is qualitatively different in two dimensions from wave propagation in three dimensions: the impulse response in two dimensions has an infinitely long tail in time.

A. Two dimensions, anti-plane case

The anti-plane case is a two dimensional elastic solid whose displacements have only one nonvanishing component, $u(x_a,t)$, $a = 1,2$, perpendicular to the (flat) equilibrium position. There is a screw dislocation that is also perpendicular to the plane, as is its Burgers vector (see Figure 1), with only one nonvanishing component b. The cut of the dislocation line by the plane is a point $X_a(x_b,t)$ [Maurel *et al.*, 2004a].

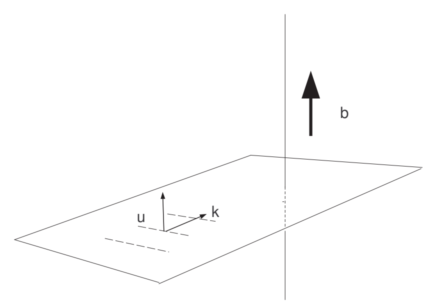

FIG. 1: Scattering of an anti-plane wave by a single screw dislocation: The motion occurs on a plane, along which the wave propagates with wave vector k. The motion u however is perpendicular to the plane, as is the Burgers vector of the screw dislocation

This is a scalar wave problem, with the dynamics of the dislocation being given by the appropriate specialization of Eqn. (2):

$$m\frac{d^2 X_a}{dt^2} = -\mu\epsilon_{ab}\frac{\partial u}{\partial x_b}(\vec{X}(t),t) \quad (8)$$

To keep things as simple as possible, we have omitted the internal friction term.

Concerning the anti-plane wave, it is described by the appropriate specialization of (5), which is

$$\rho\frac{\partial^2 v}{\partial t^2} - \mu\frac{\partial^2 v}{\partial x_a \partial x_a} = \mu b \epsilon_{ab} \dot{X}_b \frac{\partial}{\partial x_a}\delta(\vec{x} - \vec{X}(t)) \quad (9)$$

Its solution is found by the appropriate special case of Eqn. (6):

$$v(\vec{x},\omega) = \mu b \epsilon_{ab} \dot{X}_b(\omega)\frac{\partial}{\partial x_a} G^0(\vec{x},\omega) \quad (10)$$

where an overdot means time derivative and the Green's function $G^0(\vec{x},\omega)$ for the wave equation in two dimensions in infinite space with outgoing wave boundary conditions is well known to be a Hankel function of the first kind:

$$G^0(\vec{x},\omega) = \frac{i}{4\mu} H_0^{(1)}\left(\frac{\omega|\vec{x}|}{c_T}\right)$$

We are assuming small displacements, so the dislocation will move little, and the right hand side of Eqn. (8) will be evaluated at the equilibrium position of the dislocation, which we shall take to be the origin. Also, the right hand side of that equation involves particle displacement u. As mentioned above, the generation of waves by a moving dislocation is best analyzed in terms of particle velocity $v = \partial u/\partial t$. So, taking the time derivative of (8) and going to the frequency (ω) domain we obtain

$$\dot{X}_a(\omega) = -\epsilon_{ab}\frac{\mu b}{m\omega^2}\frac{\partial v}{\partial x_b}(0,\omega) \quad (11)$$

Now comes another approximation: it is assumed that the sound-dislocation interaction is sufficiently weak that the scattered wave will be small compared to the incident wave. In this case a Born approximation is appropriate and the wave in the right hand side of (11) can be taken to be the *incident* wave:

$$\dot{X}_a(\omega) = -\epsilon_{ab}\frac{\mu b}{m\omega^2}\frac{\partial v^{\rm inc}}{\partial x_b}(0,\omega) \quad (12)$$

Substitution of (12) into (10) gives the solution for the wave scattered by the screw dislocation. Taking the asymptotic behavior of the Hankel function for distances from the dislocation large compared to wavelength, we finally get the scattered wave v^s:

$$v^s(\vec{x},t) = f(\theta)\frac{e^{i\omega x/c_T}}{\sqrt{x}}e^{-i\omega t} \quad (13)$$

where we have taken an incident plane wave of frequency ω propagating along the x_1 direction:

$$v^{\rm inc}(\vec{x},t) = e^{i\omega(x_1/c_T - t)}$$

and the scattering amplitude comes out to be

$$f(\theta) = -\frac{\mu b^2}{2m}\frac{e^{i\pi/4}}{\sqrt{2\pi\omega c_T^3}}\cos\theta \quad (14)$$

with θ the angle between the direction of incidence x_1 and the direction of observation $\vec{x}$. Expression (14) was found by Eshelby [1949] and Nabarro [1951] on the basis of thinking by analogy with electromagnetism. This analogy is no longer available in the more complicated cases to be examined below. Note the, at first sight, unintuitive behavior of scattering: it grows with wavelength. This behavior does not carry over to three dimensions, where the finite length of a dislocation segment provides an additional length scale to compare wavelength with.

B. Two dimensions, in-plane case

The two dimensional in-plane case corresponds to an elastic wave propagating in two dimensions and interacting with an edge dislocation whose Burgers vector b_a lies along the plane. Particle displacements $u_a(x_b,t)$ now occur along the plane as well (see Figure 2). There are two possible polarizations for the wave: longitudinal, with particle displacements along the direction of motion, with speed of propagation c_L, and transverse, with particle displacement perpendicular to the direction of propagation, with speed of propagation c_T.

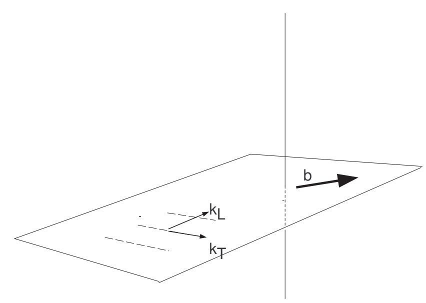

FIG. 2: Scattering of an in-plane wave by a single edge dislocation. The wave propagates along a plane, with two wave vectors: k_L for longitudinal polarization, and k_T for transverse polarization. The Burgers vector of the edge dislocation lies within the plane

We only consider dislocation glide: that is, dislocation motion along the direction of the Burgers vector. Dislocation climb would involve mass transfer and falls outside the scope of the problems for which the response of a dislocation to an external load can be discussed within the framework of continuum elasticity.

The main difference between the physics of anti-plane and in-plane wave scattering from a dislocation is mode conversion: a longitudinal incident wave can be partially scattered as a longitudinal wave, and partially as a transverse wave. Similarly, a transverse incident wave can also be partially scattered as a longitudinal wave, and partially as a transverse wave. The vector nature of the problem introduces a need for a more cumbersome notation but does not add other physical difficulties. There are thus *four* scattering amplitudes. The algebra was worked out in detail by Maurel *et al.*, [2004a] and we shall give here their expressions:

$$
\begin{aligned}
f_{LL}(\theta) &= \frac{\mu b^2}{2m}\frac{e^{i\pi/4}}{\sqrt{2\pi\omega c_L^3}}\left(\frac{c_T}{c_L}\right)^2 \sin 2\theta_0\ \sin(2\theta-2\theta_0),\\
f_{LT}(\theta) &= \frac{\mu b^2}{2m}\frac{e^{i\pi/4}}{\sqrt{2\pi\omega c_L^3}}\cos 2\theta_0\ \sin(2\theta-2\theta_0),\\
f_{TL}(\theta) &= -\frac{\mu b^2}{2m}\frac{e^{i\pi/4}}{\sqrt{2\pi\omega c_T^3}}\left(\frac{c_T}{c_L}\right)^2 \sin 2\theta_0\ \cos(2\theta-2\theta_0),\\
f_{TT}(\theta) &= -\frac{\mu b^2}{2m}\frac{e^{i\pi/4}}{\sqrt{2\pi\omega c_T^3}}\cos 2\theta_0\ \cos(2\theta-2\theta_0), \qquad (15)
\end{aligned}
$$

where the first sub index indicates the polarization of the scattered wave, and the second indicates the polarization of the incident wave. For example, f_{TL} is the amplitude for the scattering of an incident longitudinal wave into a transverse wave. The angle θ_0 is the angle between the direction of propagation of the incident (plane) wave and the Burgers vector. Note that Eqns. (15) indicate a scaling, particularly with wavelength, much the same as in the anti-plane case. Since $c_T < c_L$, both longitudinal and transverse waves have a larger amplitude for scattering into transverse waves.

C. Three dimensions, pinned dislocation segment in infinite space

We now go back to Eqn. (4) for a line segment $\vec{X}(s,t)$ of length L ($|s| \le L/2$) that represents small deviations of the position of an edge dislocation from an equilibrium position $\vec{X}_0(s,t)$ that is a straight line, with fixed end points: $\vec{X}(\pm L/2, t) = 0$ [Maurel *et al.*, 2005a].

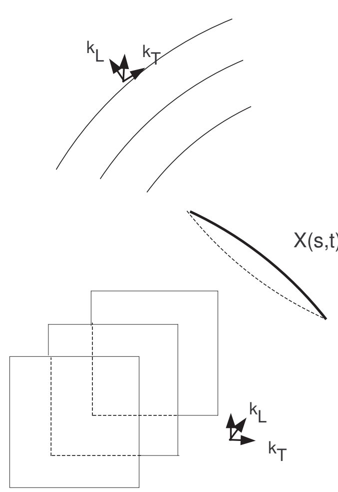

FIG. 3: An elastic plane wave is incident upon a pinned dislocation segment. The latter responds by oscillating like a string and re-radiating scattered elastic waves. Both incident as well as scattered waves have longitudinal (k_L) and transverse (k_T) polarizations

As in Section III B only glide motion is allowed so Eqn. (4) is projected along the glide direction $\hat{t}$, which is a unit vector along the direction of the Burgers vector $\vec{b} = b\hat{t}$. We shall also use a unit (binormal) vector $\hat{n} \equiv \hat{\tau} \times \hat{t}$, overdots for time derivatives and primes for derivatives with respect to s so that Eqn. (4) for $X \equiv X_k t_k$ becomes

$$
m\ddot{X}(s,t) + B\dot{X}(s,t) - \Gamma X''(s,t) = \mu b\, \mathsf{M}_{lk}\partial_l u_k(\vec{X},t), \qquad (16)
$$

with

$$
\mathsf{M}_{lk} \equiv t_l n_k + t_k n_l. \qquad (17)
$$

In this subsection we shall assume that the wavelength is large compared to the length of the dislocation segment: $kL \ll 1$. In this case it is possible to solve Eqn. (16) evaluating the Peach-Koehler force at a single point of the dislocation, say its middle point $\vec{X}_0$: $\vec{u}(\vec{X},t) \simeq \vec{u}(\vec{X}_0,t)$. In the frequency domain, this equation becomes

$$
-(m\omega^2 + i\omega B)X(s,\omega) - \Gamma X''(s,\omega) = \mu b \mathsf{M}_{lk}\partial_l u_k(\vec{X}_0,\omega), \qquad (18)
$$

whose solution, using the particle velocity $\vec{v}(\vec{x},\omega) = -i\omega\vec{u}(\vec{x},t)$, is

$$
\dot{X}(s,\omega) = -\frac{4}{\pi}\frac{\mu b}{m}\tilde{S}(s,\omega)\mathsf{M}_{lk}\partial_l v_k(\vec{X}_0,\omega), \qquad (19)
$$

with

$$
\tilde{S}(s,\omega) = \sum_{\text{odd} N} \frac{1}{N(\omega^2 - \omega_N^2 + i\omega B/m)} \sin\left[\frac{N\pi}{L}(s + L/2)\right], \qquad (20)
$$

and

$$
\omega_N \equiv \frac{N\pi}{L}\sqrt{\frac{\Gamma}{m}} \qquad (21)
$$

Taking Eqn. (6) to the frequency domain and substituting (19) into it gives

$$
\begin{aligned}
v_m^s(\vec{x},\omega) &= \frac{8L}{\pi^2}\frac{\mu b}{m}\frac{S(\omega)}{\omega^2} c_{ijkl}\ t_i\ n_j \\
&\quad \times \frac{\partial}{\partial x_l} G_{km}^0(\vec{x},\omega)\, \mathsf{M}_{np}\partial_n v_p(\vec{X}_0,\omega), \qquad (22)
\end{aligned}
$$

where we have used $\epsilon_{jnh}\dot{X}_n\tau_h = -\dot{X}n_j$ and with

$$
S(\omega) \equiv \frac{\pi\omega^2}{2L}\int_{-L/2}^{L/2} ds\ \tilde{S}(s,\omega) \simeq \frac{\omega^2}{\omega^2 - \omega_1^2 + i\omega B/m}. \qquad (23)
$$

The Green function G_{km}^0 for an infinite, homogeneous and isotropic elastic medium in three dimensions is well known [Love, 1944]. In a Born approximation, the incident wave is replaced on the right hand side of Eqn. (22) and to obtain the scattered wave the limit of G_{km}^0 for distances large (compared to both wavelength and dislocation size L) from the dislocation is used.

We shall consider an incident plane wave of frequency ω that is the superposition of a wave with longitudinal polarization, along the propagation direction $\hat{k}_L$, and a wave with linear transverse polarization along a direction $\hat{k}_T$, $\hat{k}_L \cdot \hat{k}_T = 0$, with amplitudes A_L and A_T respectively:

$$\vec{v}^{\rm inc}(\vec{x},\omega) = A_L e^{i\vec{k}_L\cdot\vec{x}}\hat{k}_L + A_T e^{i\vec{k}_T\cdot\vec{x}}\hat{k}_T \qquad (24)$$

where $k_L = \omega/c_L$ and $k_T = \omega/c_T$. Substitution of this expression into (22) leads to longitudinal, $\vec{v}^s_L$, and transverse, $\vec{v}^s_T$, scattered waves, far away from the dislocation:

$$\begin{aligned} \vec{v}^s_L(\vec{x},\omega) &= [f_{LL}A_L + f_{LT}A_T]\,\frac{e^{ik_Lx}}{x}\hat{x}, \\ \vec{v}^s_T(\vec{x},\omega) &= [f_{TL}A_L + f_{TT}A_T]\,\frac{e^{ik_Tx}}{x}\hat{y}, \end{aligned} \qquad (25)$$

with the scattering amplitudes

$$\begin{aligned} f_{LL} &= -\frac{2}{\pi^3}\left(\frac{\rho b^2}{m}\right)\gamma^{-4}\ \ L\,S(\omega)\ \ f_L(\hat{k}_0)g_L(\hat{x}) \\ f_{LT} &= -\frac{2}{\pi^3}\left(\frac{\rho b^2}{m}\right)\gamma^{-3}\ \ L\,S(\omega)\ \ f_T(\hat{k}_0)g_L(\hat{x}) \\ f_{TL} &= -\frac{2}{\pi^3}\left(\frac{\rho b^2}{m}\right)\gamma^{-1}\ \ L\,S(\omega)\ \ f_L(\hat{k}_0)g_T(\hat{x}) \\ f_{TT} &= -\frac{2}{\pi^3}\left(\frac{\rho b^2}{m}\right)\qquad L\,S(\omega)\ \ f_T(\hat{k}_0)g_T(\hat{x}) \end{aligned} \qquad (26)$$

where f_L, f_T, g_L and g_T are dimensionless functions that depend on the geometry. Note that for long wavelengths λ, all four scattering amplitudes scale like

$$f \sim L\left(\frac{L}{\lambda}\right)^2$$

so that they vanish as the ratio of dislocation length to wavelength, squared, leading to scattering cross sections inversely proportional to wavelength to the fourth power as in Rayleigh scattering. The direction of the transverse polarization is found from the geometry of the problem. Since $\gamma > 1$, both longitudinal and transverse incident waves prefer to be scattered as transverse waves, just as in two dimensions.

D. Three dimensions, pinned dislocation segment in a half space

The experiments of Shilo and Zolotoyabko, [2002; 2003] and Zolotoyabko *et al.,* [2001] provide strong motivation to tackle this problem. They have visualized the surface waves scattered by a subsurface dislocation segment in $LiNbO_3$. From a theoretical point of view, Eqn. (6) holds just as well for a half space as for a whole space, the difference being that G^0_{im} is now the impulse response function of an elastic half-space with a stress-free boundary. Finding this function is a well posed problem and has been widely studied (see Maurel *et al.* [2007a] and references therein). However, there is no expression available that compares in simplicity or ease of analytical manipulation with the expression for the impulse response for a whole space. The differences with the whole-space response are obviously especially important near the free surface. Notably, in that region, in addition to poles corresponding to the usual transverse and longitudinal waves, there is an additional singularity that corresponds to surface (Rayleigh) waves, whose speed of propagation is $c_R = \zeta c_T$, where $\zeta \sim 0.9$ is the zero of the Rayleigh polynomial $P(\zeta) = \zeta^6 - 8\zeta^4 + 8\zeta^2(3-2/\gamma^2) - 16(1-1/\gamma^2)$. These waves are not dispersive, and they have an elliptical polarization, a combination of the longitudinal and transverse perpendicular to the free surface [Landau & Lifshitz, 1981]. They penetrate the medium for a distance comparable to their wavelength. Consequently, for a dislocation to feel their presence it must be located at a distance from the free surface that is less than a few wavelengths.

In this subsection we are interested in the behavior of the secondary radiation emitted by a dislocation excited by the incident surface wave near the dislocation itself. So the concepts of scattering amplitude and cross section will not be used, they being tailored to the far-field behavior of the secondary radiation.

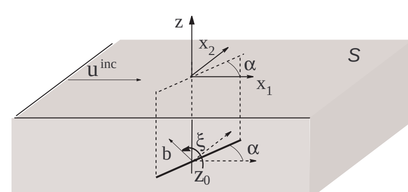

FIG. 4: A subsurface dislocation segment lying in a semi-infinite half space with a stress-free boundary is excited by a surface Rayleigh wave. It responds by oscillating, and reradiating. At the free surface, these secondary waves will interfere with the incident wave [Maurel *et al.* 2007a].

We consider the configuration of Figure 4, and the first order of business is to find the response of the dislocation to the surface wave. This is given by the solution of Eqn. (4) with the Peach-Koehler right-hand side computed from the resolved shear stress generated by the incoming surface wave at the position of the dislocation. Importantly, it is possible to find such a solution without assuming that wavelength is large compared to dislocation length. Indeed, in the case at hand the wavelength is small compared to dislocation length. The said response is a straightforward but cumbersome expression that can be found in Eqn. (3.7) of Maurel *et al.* [2007a]. In particular, we find that, for the conditions of Shilo & Zolotoyabko [2003] (wavelength $\lambda \sim 6\ \mu$m, dislocation depth of $2/k_R$) the amplitude of dislocation motion is of order 500 times the amplitude of the incident wave: $X \sim 500u^{\rm inc}_z$ which gives about 25 nm for an incident wave of amplitude 0.05 nm. The dislocation velocity $\dot{X} \sim \omega X$ thus comes out around 90 m s^{-1}, and we infer a, not unusual, drag coefficient $B \simeq 10^{-5}$ Pa s.

Once the motion of the dislocation has been found, the free surface vertical displacement $u_z^{\mathrm{s}}(\vec{x})$can be computed from Eqn. (6). As already mentioned this involves the Green function for a half space and the task is best done numerically. This result can then be superposed with the incident wave $u_z^{\mathrm{inc}}(\vec{x})$ to obtain the total displacement $u_z^{\mathrm{inc}}(\vec{x}) + u_z^{\mathrm{inc}}(\vec{x})$, as is shown on Figure 5, and compare it with the X-ray images of Shilo & Zolotoyabko [2003]. The comparison appears satisfactory [Maurel *et al.* 2007a].

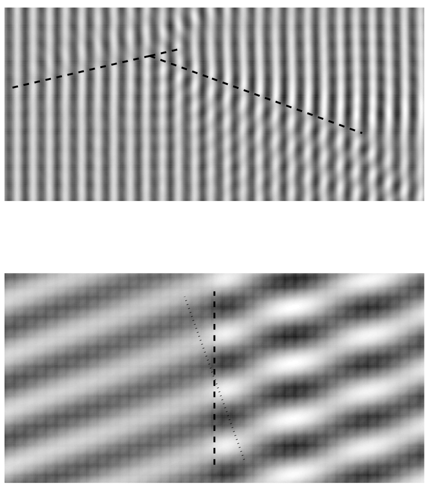

FIG. 5: The upper panel shows the interference pattern between incident and scattered waves generated at the free surface by subsurface dislocation segments whose surface trace is given by the dashed lines. The lower panel is a close up at three times the magnification of the upper panel. The dashed line is the direction of the main dislocation segment and the dotted line indicates the direction of propagation of the incident wave [Maurel *et al.* 2007a].

IV. MULTIPLE SCATTERING

Acoustic attenuation experiments are typically performed with samples that have a high dislocation density. A wave that is incident upon a dislocation-filled medium will interact with many dislocations and, in general, the resulting wave behavior will be quite complex. Under certain circumstances however, special cases may arise: the (multiply) scattered waves may interfere so as to generate a coherent, forward propagating, wave described by a complex index of refraction whose real part gives a renormalized speed of propagation, and whose imaginary part gives an attenuation (see Figure 6). The origin of this attenuation is that energy is taken away from the incident wave by the radiation that is redirected in all directions other than the incident one by the scattering: this incoherent radiation may obey, in turn, a diffusion equation characterized by a diffusion coefficient D, introducing thus a different character for energy transport. And further, it may happen that the diffusion coefficient vanish, $D = 0$, in which case, we are in the presence of Anderson localization [Sheng, 1990; 1995]. In the remaining of this paper we shall deal with coherent elastic wave propagation in an elastic continuum with many pinned dislocation segments.

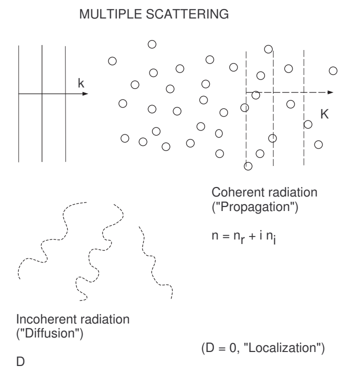

FIG. 6: An elastic plane wave is incident on a medium filled with randomly located scatterers. Part of it will propagate coherently with a complex index of refraction $n = n_r + in_i$. This implies a renormalized velocity, and an amplitude that exponentially decreases with distance. Another part of it will propagate incoherently, and may obey a diffusion equation with a diffusion constant D, whose value may, under certain circumstances, vanish, signaling Anderson localization.

When many dislocations are present, the behavior of elastic disturbances is governed by the wave equation (5) with a source term s_i that now reflects the presence of many dislocations:

$$s_i(\vec{x},t) = c_{ijkl}\epsilon_{mnk} \sum_{n=1}^{N} \int_{\mathcal{L}} ds\, \dot{X}_m^n(s,t)\tau_n b_l \times \frac{\partial}{\partial x_j}\delta(\vec{x} - \vec{X}^n(s,t)). \quad (27)$$

Replacing the solution of the string equation (4) in this source term leads to an equation of the form

$$\rho \frac{\partial^2 v_i}{\partial t^2} - c_{ijkl}\frac{\partial^2 v_k}{\partial x_j \partial x_l} = V_{ik} v_k \quad (28)$$

where

$$V_{ik} = \frac{8}{\pi^2}\frac{(\mu b)^2}{m}\frac{S(\omega)}{\omega^2} L \sum_{n=1}^{N} \mathsf{M}_{ij}^n \frac{\partial}{\partial x_j}\delta(\vec{x} - \vec{X}_0^n)\, \mathsf{M}_{lk}^n \frac{\partial}{\partial x_l}\bigg|_{\vec{x}=\vec{X}_0^n}. \quad (29)$$

Equations of this type, when V describes a random index of refraction have been much studied [Ishimaru, 1997]. Our case, however, is considerably different.

A. Two dimensions, anti-plane case

In the two-dimensional, anti-plane case, the special case of Eqn. (28) is

$$\rho\ddot{v}(\vec{x},t) - \mu\nabla^2 v(\vec{x},t) = \mu\sum_{n=1}^{N} b^n \epsilon_{ab}\dot{X}_b^n \frac{\partial}{\partial x_a}\delta(\vec{x}-\vec{X}^n). \tag{30}$$

where X^n is the solution of Eqn. (11) for the n-th dislocation. Going to the frequency domain this gives

$$\left(\nabla^2 + k_T^2\right) v(\vec{x},\omega) = -V(\vec{x},\omega)v(\vec{x},\omega), \tag{31}$$

where the "potential" V is the appropriate special case of (29):

$$V(\vec{x},\omega) = \sum_{n=1}^{N}\frac{\mu}{M}\left(\frac{b^n}{\omega}\right)^2 \frac{\partial}{\partial x_a}\delta(\vec{x}-\vec{X}_0^n)\left.\frac{\partial}{\partial x_a}\right|_{\vec{X}_0^n}. \tag{32}$$

where $k_T \equiv \omega/c_T$. In this expression, the positions $\vec{X}^n(t)$ of the screw dislocations have been replaced by their mean values (positions at rest) $\vec{X}_0^n$. The screw dislocations in the right hand side of (31) are randomly, and uniformly, placed. This is all the randomness we shall consider in this subsection.

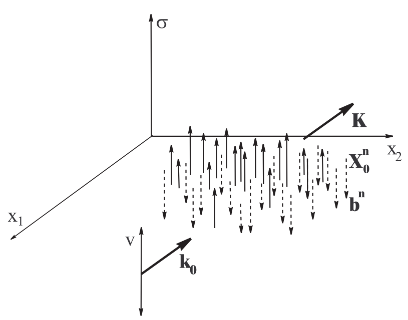

FIG. 7: Anti-plane wave propagating through a medium filled with many screw dislocations [Maurel *et al.*, 2004b]. k is the wave vector in the abscence of dislocations, and K the effective wave vector for coherent propagation.

In the absence of scatterers, plane waves propagate with a wave vector k_T. The exercise is to find an effective wave vector K_T, if it exists, for the propagation of coherent plane waves.

A result that goes back at least to Foldy [1945] states that, for low densities n of scatterers, this many-body problem can be reduced to a single body one:

$$K_T = k_T + n\sqrt{\frac{2\pi}{k}}\langle f(0)\rangle e^{-i\pi/4} \tag{33}$$

where the effective wave vector is given in terms of the forward scattering amplitude for single-body scattering, and the brackets denote an average over all internal degrees of freedom (absent in the present case). We already know the scattering amplitude, Eqn. (14), so it is easy to replace in (33) to get

$$K_T = k_T\left(1 - \frac{\mu n b^2}{2m\omega^2}\right). \tag{34}$$

A simple minded expression for the attenuation length Λ is given by

$$\Lambda^{-1} = n\sigma$$

where n is the dislocation density and σ is the total scattering cross section for single dislocation scattering defined as:

$$\sigma = \int_0^{2\pi} d\theta |f(\theta)|^2$$

Since we know the scattering amplitude for all angles, $f(\theta)$ Eqn. (14), not only in the forward direction, we can compute the total cross section to get an expression for the attenuation length:

$$\Lambda = \frac{8m^2 c_T^4 k}{n\mu^2 b^4}. \tag{35}$$

This simple minded approach is not available in more complicated situations so we describe a more systematic approach, based on Green functions [Maurel *et al.*, 2004b].

The Green function approach is based on the fact that, in the case when no scatterers are present, the propagation character of the wave is determined by the poles of the corresponding Green function $G^0(\vec{k},\omega)$ in wavenumber space:

$$G^0(\vec{k},\omega) = \frac{1}{k_T^2 - k^2}$$

The strategy provided by Multiple Scattering theory is to focus on the impulse response function G of the medium filled with scatterers, in our case screw dislocations:

$$\left(\nabla^2 + k_T^2 + V(\vec{x})\right)G(\vec{x},\vec{x}') = -\delta(\vec{x}-\vec{x}'). \tag{36}$$

Then, since this equation involves a random operator V, consider the average $< G(\vec{k},\omega) >$, and look for its poles in the form

$$\left\langle G(\vec{k},\omega)\right\rangle = \frac{1}{k_T^2 - \Sigma(k) - k^2} \tag{37}$$

where $\Sigma(k)$, the mass operator, is a functional of the "potential" V. Assuming (a big assumption) that there will be a coherent wave whose (effective) wave vector does not differ very much from the wave vector in the absence of scatterers, the problem becomes finding the mass operator Σ perturbatively, treating the "potential" V as the perturbation. To second order in V, the result is

$$\Sigma = \langle V\rangle + \langle VG^0V\rangle - \langle V\rangle G^0\langle V\rangle. \tag{38}$$

The result of this calculation in the current, antiplane case, is

$$\Sigma(k) = \frac{\mu n b^2}{m\omega^2} k^2 \left[-1 + \frac{i\rho b^2}{8m} \right]. \quad (39)$$

to leading order in the real part, as well as the imaginary part, of the mass operator. This expression can be readily generalized [Maurel *et al.*, 2004b] to the case when the dislocations have random Burgers vectors, as well as random masses.

We are now in a position to obtain the effective wave number K_T, the pole of $\langle G \rangle$ in Eqn. (37). Since K_T is supposed to be close to k_T we get

$$\begin{aligned} K_T &\simeq k_T \left(1 + \frac{\Sigma(k_T)}{2k_T^2}\right), \quad (40) \\ &\simeq k_T \left(1 + \frac{\mu n b^2}{2m\omega^2}\left[-1 + \frac{i\rho b^2}{8m}\right]\right). \end{aligned}$$

The real part of this expression coincides with (34), and the imaginary part gives the attenuation length (35) through:

$$\Lambda^{-1} = 2\Im(K_T)$$

The effective velocity $v_{\text{eff}} \equiv \omega/\Re(K_T)$ for coherent wave propagation is thus

$$v_{\text{eff}} = c_T \left(1 + \frac{\mu n b^2}{2m\omega^2}\right)$$

Note that the effective phase velocity is greater than c_T, although the group velocity is smaller. This is probably an effect of the bidimensionality, and just like the dependence on wavelength, it does not carry over to three dimensions.

B. Two dimensions, in-plane case

As already mentioned, the main difference between the anti-plane and in-plane cases is that the latter is of a vector nature: a wave with both longitudinal and transverse polarizations travels in a medium filled with randomly placed edge dislocations, whose Burgers vectors are randomly oriented (see Figure 8. There will be two effective velocities and two attenuation lengths, one for each polarization. On the technical side, the results of Foldy, and of the Green function approach, mentioned in the previous subsection, have to be generalized to this vector case. The "potential" V, the Green function G and mass operator Σ become rank-two matrices, and the effective wave vectors are given by the zeros of the determinant of $< G >^{-1}$.

The generalization of the simple-minded approach of

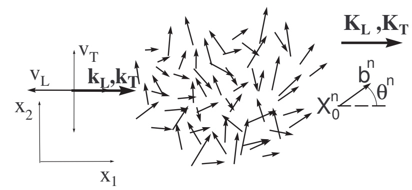

FIG. 8: An inplane wave is incident on a medium with many edge dislocations whose Burgers vectors are randomly oriented. k_L and k_T are the longitudinal and transverse wave vectors, respectively, in the absence of dislocations. K_L and K_T are the effective longitudinal and transverse wave vectors of the coherently propagating wave [Maurel *et al.*, 2004b].

Foldy to the anti-plane case gives [Maurel *et al.*, 2004b]

$$\begin{aligned} K_c &= k_c + n\sqrt{\frac{2\pi}{k_c}} \langle f_{cc} \rangle_{b,\theta}(0) e^{-i\pi/4}, \\ &= k_c \left(1 - \frac{n\mu b^2 \mathcal{A}_c}{4m\omega^2}\right), \quad (41) \end{aligned}$$

with

$$\begin{aligned} \mathcal{A}_L &= \gamma^{-2}, \\ \mathcal{A}_T &= 1. \quad (42) \end{aligned}$$

There is no mode conversion into the forward direction through scattering. The brackets $\langle\rangle_{b,\theta}$ mean taking the average for the magnitudes and orientations of the Burgers vectors. The second line in Eqn. (41) is obtained when all Burgers vectors have the same magnitude and all orientations are equally likely.

The generalization of the Green function method leads to

$$K_c = k_c \left\{ 1 - \frac{\mu n b^2 \mathcal{A}_c}{4m\omega^2} \left[1 - \left(1 + \frac{1}{\gamma^4}\right) \frac{i\rho b^2}{8m} \right] \right\}, \quad (43)$$

The real part of this expression coincides with (41) and the imaginary part gives two attenuation lengths, one for longitudinal and one for transverse wave propagation

$$\begin{aligned} \Lambda_L &= 16 \frac{m^2}{n\rho^2 b^4} \frac{\gamma^8}{\gamma^4 + 1} k_L. \\ \Lambda_T &= 16 \frac{m^2}{n\rho^2 b^4} \frac{\gamma^4}{\gamma^4 + 1} k_T. \end{aligned} \quad (44)$$

The above expression for attenuation length also satisfy

$$\begin{aligned} \Lambda_L^{-1} &= n(\sigma_{LL} + \sigma_{LT}) \\ \Lambda_T^{-1} &= n(\sigma_{TL} + \sigma_{TT}) \quad (45) \end{aligned}$$

where the scattering cross sections σ_{ab} are given in terms of the scattering amplitudes (15) by

$$\sigma_{ab} = \int_0^{2\pi} d\theta < |f_{ab}|^2 > \quad (46)$$

The effective velocities $v^c_{\rm eff} \equiv \omega/\Re(K_c)$ for coherent wave propagation are easily read off from (43):

$$v^c_{\rm eff} = c_c\left(1+\frac{\mu n b^2}{4m\omega^2}\mathcal{A}_c\right).$$

Just as in the antiplane case, the effective phase velocity, but not the effective group velocity, is larger in the presence than in the absence of dislocations, an effect probably due to the bidimensionality of the situation.

C. Three dimensions, many dislocation segments

We now study the configuration illustrated on Fig. 9 [Maurel *et al.,* 2005b]. A plane elastic wave propagates through an elastic medium that is filled with edge dislocation segments whose end points are fixed, and they are straight lines in equilibrium. The position of these segments is random, with a uniform distribution through space. The orientation of the lines is also random, also uniformly distributed throughout the solid angle. All segments have the same length L and magnitude of Burgers vector b. The latter is perpendicular to the line, but, within that plane, the orientation is random, uniformly distributed in angle. Dropping the assumptions of uniform distribution for the orientation of the lines and of the Burgers vectors does not lead to significant complications. Dropping the assumption of constant L, however, would lead to significant changes, as would dropping the assumption of uniform distribution in space. Throughout this subsection, wavelengths will be assumed to be large compared to dislocation line segment length L.

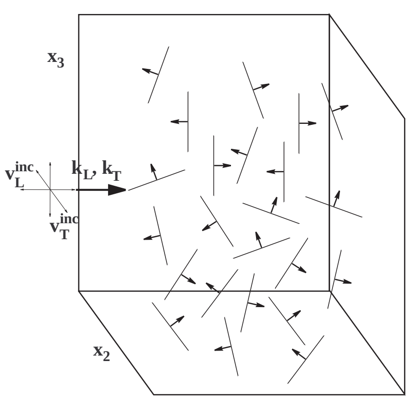

FIG. 9: An elastic wave with both longitudinal (L) and transverse (T) polarizations is incident upon a medium filled with many, randomly placed and oriented, edge dislocations segments whose Burgers vector are randomly oriented as well. [Maurel *et al.,* 2005b].

In the Green function approach to multiple scattering for this system, the starting point for the algebra is the set of equations

$$\rho\omega^2 G^0_{im}(\vec{x},\omega)+c_{ijkl}\frac{\partial^2}{\partial x_j\partial x_l}G^0_{km}(\vec{x},\omega) = -\delta_{im}\delta(\vec{x}) \quad (47)$$

for the rank-three Green tensor G^0_{im} that describes the impulse response function of the elastic medium in the absence of dislocations, and

$$\rho\omega^2 G_{im}(\vec{x},\omega) \;+\; c_{ijkl}\frac{\partial^2}{\partial x_j\partial x_l}G_{km}(\vec{x},\omega) = -V_{ik}G_{km}(\vec{x},\omega)-\delta_{im}\delta(\vec{x}), \quad (48)$$

where V is the "potential" of Eqn. (28), for the rank-three Green tensor G_{im} that describes the impulse response function of the elastic medium with a random distribution of dislocations. Both equations have been written in the frequency domain.

The exercise now is, as in two dimensions, to find effective wave numbers K_a ($a = L, T$) for longitudinal as well as transverse coherent wave propagation. In general these effective wave numbers will be complex, with the real part leading to effective speeds of wave propagation, and the imaginary part leading to attenuation lengths. They are found as the zeroes of the determinant of the three by three matrix for the inverse of the averaged Green tensor G in wave number space:

$$< G(\vec{k},\omega) >^{-1} = G^0(\vec{k},\omega)^{-1} - \Sigma(\vec{k},\omega) \quad (49)$$

where the mass operator Σ is a functional of the "potential" V that can be found perturbatively. The result is, to leading order for both the real and imaginary parts

$$\begin{aligned} K_L &\simeq k_L\left[1-\frac{16}{15\pi^2}\frac{1}{\gamma^4}\frac{\rho b^2}{m}\frac{S(\omega)}{\omega^2}nLc_L^2 \right.\\ &\left. +i\frac{64}{225\pi^5}\frac{3\gamma^5+2}{\gamma^8}\left(\frac{\rho b^2}{m}\right)^2\frac{\Re\left[S^2(\omega)\right]}{\omega}nL^2c_L\right],\\ K_T &\simeq k_T\left[1-\frac{4}{5\pi^2}\frac{\rho b^2}{m}\frac{S(\omega)}{\omega^2}nLc_T^2 \right.\\ &\left. +i\frac{16}{75\pi^5}\frac{3\gamma^5+2}{\gamma^5}\left(\frac{\rho b^2}{m}\right)^2\frac{\Re\left[S^2(\omega)\right]}{\omega}nL^2c_T\right] \end{aligned} \quad (50)$$

where $S(\omega)$ is given by (23) and n is the number of dislocation segments per unit volume. The magnitude of these effective wavenumbers is independent of orientation since the effective medium is isotropic. This is a consequence that, whenever random vector quantities appear, all directions are equally likely.

The real part of these expressions leads to effective wave velocities $v_a = \omega/\Re(K_a)$ that are straightforward to find. In the long wavelength limit they are

$$\begin{aligned} v_L &\simeq c_L\left(1-\frac{16}{15\pi^4}\frac{1}{\gamma^2}\frac{\mu b^2}{\Gamma}\frac{nL^3}{1+(\omega/\omega_B)^2}\right)\\ v_T &\simeq c_T\left(1-\frac{4}{5\pi^4}\frac{\mu b^2}{\Gamma}\frac{nL^3}{1+(\omega/\omega_B)^2}\right), \end{aligned} \quad (51)$$

where $\omega_B = m\omega_1^2/B$, and ω_1 is the fundamental mode for oscillations of the string-like dislocation segment, given by (21), the only mode of importance for long wave lengths.

The imaginary part leads similarly to attenuation coefficients $\alpha_a = \Im(K_a)$:

$$\begin{aligned}\alpha_L &= \frac{16}{15\pi^6}\frac{1}{\gamma^2}\frac{\mu b^2 nL^5}{c_L\Gamma^2}\left(\frac{B\omega^2}{1+(\omega/\omega_B)^2}\right. \\ &\quad \left. +\frac{4(3\gamma^5+2)}{15\pi^3\gamma^4}\frac{\rho b^2 L}{c_L}\frac{\omega^4[1-(\omega/\omega_B)^2]}{[1+(\omega/\omega_B)^2]^2}\right), \\ \alpha_T &= \frac{4}{5\pi^6}\frac{\mu b^2 nL^5}{c_T\Gamma^2}\left(\frac{B\omega^2}{1+(\omega/\omega_B)^2}\right. \\ &\quad \left. +\frac{4(3\gamma^5+2)}{15\pi^3\gamma^5}\frac{\rho b^2 L}{c_T}\frac{\omega^4[1-(\omega/\omega_B)^2]}{[1+(\omega/\omega_B)^2]^2}\right). \end{aligned} \tag{52}$$

Note that, for low frequencies and low damping, $\omega \ll \omega_B$, both damping coefficients scale with frequency as

$$\alpha \sim C_1(B)\omega^2 + C_2\omega^4 \tag{53}$$

This is a reflection of the fact that the damping of the coherent wave has two origins: internal losses, scaling like frequency squared, and originated by disorder, scaling like frequency to the fourth.

Finally, for fixed total dislocation line density, that is fixed nL, measured in units of inverse surface, attenuation coefficients scale with segment length L as

$$\alpha \sim L^4$$

and the velocity changes $\Delta v = c - v$ scale as

$$\frac{\Delta v}{c} \sim L^2$$

just as in the GL theory [Granato & Lücke, 1966]. More on this below.

1. *Comparison of current theory with attenuation experiments*

The results developed in the previous paragraphs can be put to the test since measurements of longitudinal and transverse wave attenuation are available. Indeed, quality factors Q have been measured for copper single crystals [Ogi *et al.*, 1999], copper polycrystals [Ledbetter, 1995] and $LiNbO_3$ [Ogi *et al.,* 2004] using resonant ultrasound spectroscopy (RUS), and its electromagnetic variant (EMAR) for copper single crystals [Ogi *et al.*, 1999]. The frequency ranges of these experiments fall into what has been called "low frequencies" in the calculations above, and it is a simple matter to find the ratio of the longitudinal to transverse quality factors:

$$\frac{Q_T^{-1}}{Q_L^{-1}} \simeq \frac{3}{4}\gamma \sim 1.30 - 1.56, \tag{54}$$

	Q_L/Q_T
Polycrystallyne copper using RUS (a)	1.6
$LiNbO_3$ using RUS (b)	1.7
Copper single crystals using RUS (c)	1.4
Copper single crystals using EMAR (c)	2.1-2.3
Current theory	1.3-1.6

TABLE I: The ratio of longitudinal to transverse wave attenuation according to current theory (bottom line) and according to experiments: (a) Ledbetter & Fortunko [1995]; (b) Ogi *et al.*, [2004]; (c) Ogi *et al.,* [1999]

Note that γ is just the ratio of longitudinal to transverse wave propagation velocity, a quantity determined completely by the material at hand. In this ratio there is no dependence on material density, elastic constants, or dislocation paramenters such as mass, line tension or attenuation coefficient. There are thus no adjustable parameters. The range of values given in Eqn. (54) reflects a range in Poisson's ratio of 0.25 to 0.35. Table I compares the data with this value. The agreement is satisfactory.

2. *Granato-Lücke theory as a special case of current theory*

The theory of Granato & Lücke [1966] is a special case of the theory developed in the present paragraph. Indeed, its starting point is the set of equations

$$\begin{aligned}\rho\frac{\partial^2}{\partial t^2}\sigma(\vec{x},t) - \mu\frac{\partial^2}{\partial x_2^2}\sigma(\vec{x},t) &= -\mu\rho\, b\frac{\Lambda}{L}\int ds\,\ddot{\xi}(\vec{x},s,t), \\ m\ddot{\xi} + B\dot{\xi} - \Gamma\xi'' &= \mu\sigma.\end{aligned} \tag{55}$$

where Λ is the total length of movable dislocation ($\Lambda = nL$ in our notation), ξ is a scalar string (a simplification of our $\vec{X}$), and σ is a scalar stress (a simplification of our σ_{ij}). This system is solved looking for solutions of the form

$$\sigma(x_2,t) = \sigma_0 e^{-\alpha_{GL}x_2}e^{i\omega(t-x_2/v_{GL})}, \tag{56}$$

that is, attenuated waves traveling along the x_2 direction, where α_{GL} and v_{GL} are, respectively the attenuation and the effective wave velocity in the GL model, and it is assumed that ξ does not depend on x_3.

It is a straightforward exercise to check that Eqns. (55) are obtained within our multiple scattering formalism when looking at the system of Figure 10. That is, a purely transverse waves is incident on an ensemble of dislocation segments randomly placed but not randomly oriented: they must all be parallel, along the x_3 direction, according to Eqn. (55). Also, all the Burgers vectors must point along the same, x_1, direction. In the long wavelength limit, attenuation length and effective wave velocity obtained from this calculation coincides with our

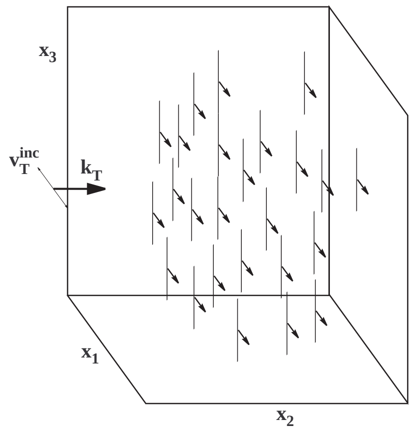

FIG. 10: Applying a multiple scattering formalism to a shear wave incident upon a medium filled with many, randomly placed but parallel, dislocation segments reproduces the Granato-Lücke theory [Maurel *et al.,* 2005b].

own, up to numerical coefficients. The distinguishing feature of our theory is that it can clearly discriminate between longitudinal and transverse polarizations. Also, the effective medium corresponding to the situation depicted in Figure 10 is anisotropic so that an effective wave vector will depend on the direction of propagation.

D. Three dimensions, low angle grain boundaries

It is well known that low angle grain boundaries can be modeled as arrays of dislocations [Bragg, 1940; Burgers, 1940; Read & Schockley, 1950; Shockley & Read, 1949], see Figure 11.

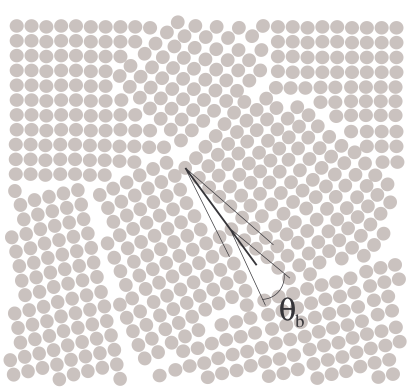

FIG. 11: Low angle grain boundaries can be considered as arrays of dislocations [Maurel *et al.,* 2006].

On the other hand, Zhang et al. [2004] have measured the frequency dependence of acoustic attenuation in polycrystalline copper with enough accuracy to find that it scales as a linear combination of quadratic and quartic terms:

$$\alpha \sim c_1\omega^2 + c_2\omega^4. \tag{57}$$

While there appears to be general agreement that the quartic term arises from Rayleigh scattering by the grain boundaries, the quadratic term is a surprise. Is it possible to rationalize these findings within a multiple scattering framework?

We have already seen that this type of behavior is what happens, for long wavelengths, when a coherent wave propagates in a medium filled with many dislocations. The quadratic term is associated with internal losses, and the quartic term is associated with losses due to the disorder: The multiple scattering takes energy away from the forward-moving coherent wave.

To proceed, the many dislocations considered in Eqn. (28) are distributed in two classes: one class is grouped as a number of parallel segments, that mimic a low angle grain boundary. Then, each such class is randomly distributed throughout space, see Figure 12. The details in two dimensions can be found in Maurel *et al.* [2006], and, in three dimensions, in Maurel *et al.* [2007b].

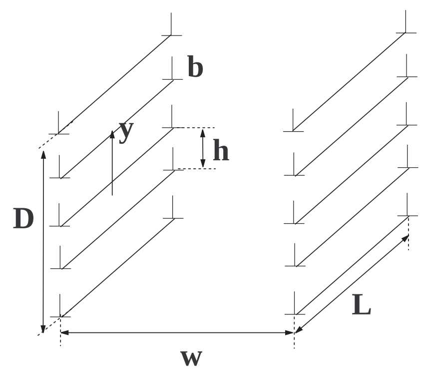

FIG. 12: Dislocation segments grouped in two classes mimic, at long wavelengths, the response of randomly oriented grain boundaries to acoustic waves [Maurel *et al.*, 2007b]. Here, w would be the grain size, D and L the rectangular grain lengths, and h the spacing between dislocations within a grain boundary.

This modeling does not capture the topology of the grain boundaries, in the sense that such walls must meet at joints, to actually enclose grains. However, long wavelength propagation should be independent of these characteristics. The result is [Maurel *et al.,* 2007b] that the prefactor of the quartic term in Eqn. (57) can be obtained with reasonable values for the material under study, without adjustable parameters. The usual source of attenuation in polycrystals, changes in the elastic constants from grain to grain, can be incorporated as an additive effect, in perturbation theory. The prefactor of the quadratic term can be fit assuming that the drag on the dynamics of the dislocations making up the wall is one to two orders of magnitude smaller than the value usually accepted for isolated dislocations. As mentioned in Section II, the expressions (3) for mass and line tension are valid for isolated dislocations. More precisely, they involve a cut off that is the distance to the nearest defect. In the case at hand, this distance is h, small compared to wavelength, and these values should be revisited for a more accurate assessment.

V. CONCLUSIONS AND OUTLOOK

We have provided a fairly complete theory of the interaction of elastic waves with dislocations, both in isolation and in large numbers, leading to the understanding of a number of quantitative experimental results, as well as of a number of visualizations. What next?

As stated in the Introduction, one motivation for this work has been to develop non intrusive probes of plastic behavior of materials. The next logical step might be to validate the results obtained with the current theory with those obtained using conventional techniques. For example two identical materials differing only in their density of dislocations, a situation that appears not impossible to achieve, would give rise to different speeds of sound, leading to different resonant frequencies in identically-shaped samples. The dislocation density so inferred could then be compared with the density measured through Transmission Electron Microscopy.

Assuming that step to be successful, the next step would be to develop suitable, hopefully portable instrumentation to probe dislocation density in any sample of interest. The obstacle to be overcome would be to differentiate the contribution of dislocations from the contribution of other scatterers such as inclusions, vacancies, grain boundaries and so on. This does not look an insurmountable job given the specific nature, such as angular dependence, of the radiation scattered by dislocations compared to other defects.

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References

Anderson, A. C. [1983], in *Dislocations in Solids*, edited by F. R. N. Nabarro, Ch. 29 (North Holland).

Arakawa, K., Hatakana, M., Kuramoto, E., Ono, K. & H. Mori, H. [2006], "Changes in the Burgers Vector of Perfect Dislocation Loops without Contact with the External Dislocations", *Phys. Rev. Lett.* **96**, 125506.

J.M. Burgers [1940], "Geometrical considerations concerning the structural irregularities to be assumed in a crystal", *Proc. Phys. Soc.* **52**, 23-33.

Bragg, W.L. [1940], "The structure of a cold-worked metal", *Proc. Phys. Soc. Lond.* **52**, 105-109.

Eshelby, J. D. [1949], "Dislocation as a cause of mechanical damping in metals", *Proc. Roy. Soc. A.* **197**, 396-416.

Foldy, L. L. [1945], "The multiple scattering of waves: General theory of isotropic scattering by randomly distributed scatterers", *Phys. Rev.* **67**, 107-119.

Granato, A. V., & Lücke, K. [1956a], "Theory of Mechanical damping due to dislocations", *J. Appl. Phys.* **27**, 583-593.

Granato, A. V., & Lücke, K. [1956b], "Application of dislocation theory to internal friction phenomena at high frequencies", *J. Appl. Phys.* **27**, 789-805.

Granato, A. V., & Lücke, K. [1966], in *Physical Acoustics*, Vol 4A, edited by W. P. Mason (Academic).

Ishimaru, A. [1997], *Wave Propagation and Scattering in Random Media* (IEEE/OUP).

Kenderian, S., Berndt, T. P., Green Jr., R. E. & Djordjevic, B. B. [2003a], "Ultrasonic monitoring of dislocations during fatigue of pearlitic rail steel", *Mat. Sci. & Eng. A* **348**, 90-99.

Kenderian, S., T. P. Berndt, T. P, R.E. Green Jr., R. E., & Djordjevic, B. B. [2003b], "Ultrasonic attenuation and velocity in pearlitic rail steel during fatigue using longitudinal wave probing" , *J. Test. Eval.* **31**, 98-105.

Koehler, J. S. [1952], in *Imperfections in nearly Perfect Crystals*, edited by W. Schockley *et al.* (Wiley).

Landau, L. D. & Lifshitz, E. M. [1981], *Theory of Elasticity*, 2nd edition (Pergamon).

Ledbetter, H. M. & Fortunko, C. [1995], "Elastic constants and internal friction of polycrystalline copper", *J. Mater. Res.* **10**, 1352-1353.

Love, A. E. H. [1944], *The Mathematical Theory of Elasticity* (Dover).

Lund, F. [1988], "Response of a stringlike dislocation loop to an external stress", *J. Mat. Res.* **3**, 280-297.

Maurel, A., Mercier, J.-F., & Lund, F. [2004a], "Scattering of an elastic wave by a single dislocation", *J. Acoust. Soc. Am.* **115**, 2773-2780.

Maurel, A., Mercier, J.-F. & Lund, F. [2004b], "Elastic wave propagation through a random array of dislocations", *Phys. Rev. B* **70**, 024303.

Maurel, A., Pagneux, V., Barra, F. & Lund, F. [2005a] "Interaction between an elastic wave and a single pinned dislocation" *Phys. Rev. B* **72**, 174110.

Maurel, A., Pagneux, V., Barra, F. & Lund, F. [2005b] "Wave propagation through a random array of pinned dislocations: Velocity change and attenuation in a generalized Granato and Lücke theory", *Phys. Rev. B* **72**, 174111.

Maurel, A., Pagneux, V., Boyer, D., & Lund, F. [2006] "Propagation of elastic waves through polycrystals: the effects of scattering from dislocation arrays", *Proc. R. Soc. Lond. A*, **462**, 2607-2623.

Maurel, A., Pagneux, V., Barra, D. & Lund, F. [2007a], "Interaction of a Surface Wave with a Dislocation", *Phys. Rev. B* **75**, 224112.

Maurel, A., Pagneux, V., Barra, D. & Lund, F. [2007b] "Multiple scattering from assemblies of dislocation walls in three dimensions. Application to propagation in polycrystals", *J. Acoust. Soc. Am.*, **121**, 3418-3431.

Mura, T. [1963] "Continuous distribution of moving dislocations", *Phil. Mag.* **8**, 843-857.

Nabarro, F. R. N. [1951], "The interaction of screw dislocations and sound wave", *Proc. Roy. Soc. A.* **209**, 278-290.

Ogi, H., Ledbetter, H.M., Kim, S. & Hirao, M. [1999], "Contactless mode-selective resonance ultrasound spectroscopy: Electromagnetic acoustic resonance", *J. Acoust. Soc. Am.* **106**, 660-665.

Ogi, H., Nakamura, N., Hirao, M. & Ledbetter, H. [2004], "Determination of elastic, anelastic, and piezoelectric coefficients of piezoelectric materials from a single specimen by acoustic resonance spectroscopy", *Ultrasonics* **42**, 183-187.

Ohtani, T., Ogi, H. & Hirao, M. [2005], "Acoustic damping characterization and microstructure evolution in nickel-based superalloy during creep", *Int. J. Sol. and Struct.* **42**, 2911-2928.

Peach, M. O., & Koehler, J. S. [1950], "The Forces Exerted on Dislocations and the Stress Fields Produced by them", *Phys. Rev.*, **80**, 436-439.

Read, W.T., & Shockley, W. [1950], "Dislocation models of crystal grain boundaries", *Phys. Rev.* **78**, 275-289.

Roberts, S. G. [2000], in *Multiscale Phenomena in Plasticity*, edited by J. Lepinoux *et al.* (Kluwer).

Robertson, I. A., Ferreira, P. J., Dehm, G., Hull, R. & Stach, E. A. [2008] "Visualizing the Behavior of Dislocations—Seeing is Believing", *MRS Bulletin* **33**, 122-131.

Sheng, P. [1990], *Scattering and Localization of Classical Waves in Random Media* (World Science).

Sheng, P. [1995], *Introduction to Wave Scattering, Localization, and Mesoscopic Phenomena* (Academic Press).

Shilo, D., & Zolotoyabko, E. [2002], "Visualization of surface acoustic wave scattering by dislocations" *Ultrasonics* **40**, 921-925.

Shilo, D., & Zolotoyabko, E. [2003], "Stroboscopic x-ray imaging of vibrating dislocations excited by 0.58 GHz phonons", *Phys. Rev. Lett.* **91**, 115506.

Shockley, W., & Read, W. T. [1949], "Quantitative predictions from dislocation models of crystal grain boundaries", *Phys. Rev.* **75**, 692.

Suresh, S. [1998], *Fatigue of Materials*, 2nd. Edition (Cambridge University Press).

Xu, X., Beckman, S. P., Specht, P., Weber, E. R., Chrzan, D. C., Erni, R. P., Arslan, I., Browning, N., Bleloch, A. & Kisielowski, C. [2005] "Distortion and Segregation in a Dislocation Core region at Atomic Resolution" *Phys. Rev. Lett.* **95**, 145501.

Zhang, X.-G., Simpson Jr., W. A., & Vitek, J.M., [2004] "Ultrasonic attenuation due to grain boundary scattering in copper and copper-aluminium", *J. Acoust. Soc. Am.* **116**, 109-116.

Zolotoyabko, E., Shilo, D. & Lakin, E. [2001], "X-ray imaging of acoustic wave interaction with dislocations" *Mater. Sci. Eng. A* **309**, 23-27.