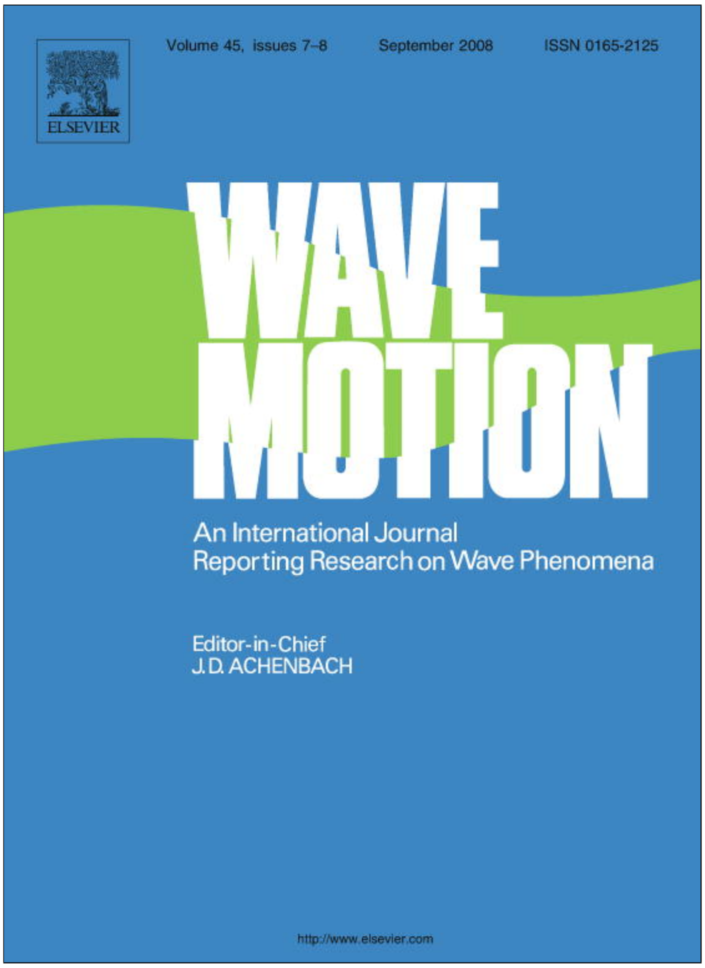
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Multiple scattering by random configurations of circular cylinders: Weak scattering without closure assumptions

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ABSTRACT

Acoustic scattering by random collections of identical circular cylinders is considered. Each cylinder is penetrable, with a sound-speed that is close to that in the exterior: the scattering is said to be "weak". Two classes of methods are used. The first is usually associated with the names of Foldy and Lax. Such methods require a "closure assumption", in addition to the governing equations. The second class is based on iterative approximations to integral equations of Lippmann–Schwinger type. Such methods do not use a closure approximation. Our main result is that both approaches lead to exactly the same formulas for the effective wavenumber, correct to second-order in scattering strength and second-order in filling fraction. Approximations for the average wavefield are also derived and compared.

1. Introduction

Understanding multiple scattering by random arrays of identical obstacles is a problem of long-standing interest. One approach, going back to Foldy [1], begins with a deterministic model for scattering by N obstacles, which we write concisely as

$$u = u_{\text{in}} + \sum_{n=1}^{N} \mathscr{K}_n u_n, \tag{1.1}$$

where u is the unknown wavefield, u_{in} is the given incident field and $\mathscr{K}_n$ is an operator. The quantity u_n can be viewed as the (unknown) contribution to u coming from the scatterer centred at $\mathbf{r}_n$. Then, the ensemble average, $\langle u \rangle$, is calculated over all possible configurations of the scatterers. It follows from the right-hand side of Eq. (1.1) that we have to calculate $\langle \mathscr{K}_n u_n \rangle$, but this only exists if there is a scatterer at $\mathbf{r}_n$: we are forced to introduce a conditional average of u. Thus, we cannot derive an equation for $\langle u \rangle$, merely a hierarchy of equations relating various different conditional averages of u. Breaking this hierarchy requires an additional "closure assumption". Such assumptions are difficult to justify, in general; for an example, see [2].

For an alternative approach, suppose that we had an explicit formula for u, for any given configuration of the scatterers. Then, we could calculate $\langle u \rangle$ directly, in principle, without the use of closure assumptions. Of course, we do not have such an explicit formula for u, but we do have explicit *approximations* for u, and these could be used. This idea also has a long history (see, for example, [3] and [4, §7.4.2]).

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In this paper, we shall compare these two approaches for a simple two-dimensional, time-harmonic, acoustic problem, where detailed explicit calculations can be made. We choose random arrays of identical penetrable circles. The number of circles per unit area is n_0. We assume that the area fraction occupied by the circles, $\phi \equiv \pi a^2 n_0$, is small, where each circle has radius a. We also assume that the scattering is "weak", meaning that the strength $m_0 = 1 - (k_0/k)^2$ is small, where k and k_0 are the wavenumbers in the exterior and interior, respectively. The assumption of small n_0 is common in many theoretical studies (the scatterers are dilute), whereas the assumption of small m_0 is convenient for the derivation of good approximations to the deterministic problem. We work to second-order in both m_0 and n_0. Maurel [5] has made similar comparisons, but with uncorrelated "point scatterers", meaning that each scatterer is represented by a Dirac delta function; we are interested here in scatterers of finite size (and we do not make small-ka approximations). Comparisons with Maurel's results are made in Section 5.3.

One output from the analysis is a formula for the effective wavenumber, K: the averaged field $\langle u \rangle$ is found to solve the Helmholtz equation, $(\nabla^2 + K^2)\langle u \rangle = 0$, in a fictitious "effective medium". For small n_0, formulas for K look like

$$K^2 \simeq k^2 + n_0\delta_1 + n_0^2\delta_2, \tag{1.2}$$

with explicit expressions for δ_1 and δ_2. (For references to the literature, see [6] and [7].) Foldy-type theories use his closure assumption, they are essentially linear in n_0, they assume that the scatterers are independent, and they predict that

$$K^2 = k^2 - 4in_0 f(0), \tag{1.3}$$

where f is the far-field pattern for one scatterer in isolation (see Eq. (2.7)). Also, when $u_{in} = e^{ikx}$,

$$\langle u \rangle = Ae^{iKx}, \tag{1.4}$$

where Foldy theory predicts that the "amplitude", A, is given by

$$A = 1 + in_0 k^{-2} f(0). \tag{1.5}$$

Observe that if we write $A = |A|e^{i\alpha}$ and $K = K_r + iK_i$, then

$$\mathrm{Re}\{\langle u \rangle e^{-i\omega t}\} = |A|e^{-K_i x}\cos(K_r x - \omega t + \alpha), \tag{1.6}$$

showing that the quantities $|A|$ and $K_i = \mathrm{Im}\,K$ are of interest.

For one circular scatterer, we can construct the exact solution by separation of variables. This elementary calculation is outlined in Section 2; it yields an exact formula for $f(0)$, which can then be inserted in Eqs. (1.3) and (1.5).

For our model problem, the fluid density is constant everywhere, both inside and outside the scatterers. Consequently, the scattering problem can be solved using the Lippmann–Schwinger integral equation (see Section 3). (This equation can also be used when k_0 is a function of position; if the density inside differs from that outside, a different integral equation must be used [8].) Under certain circumstances, the Lippmann–Schwinger equation can be solved by iteration; we examine the first-order (Born) approximation (linear in the strength m_0) and the second-order approximation (quadratic in m_0). For one circular scatterer, this is done in Section 4. We confirm that the second-order iterative approximation for the wavefield (both inside and outside the scatterer) agrees with the second-order approximation of the exact solution.

Scattering by random arrangements of scatterers is considered in Section 5. We write down the iterative approximation for scattering by N circles. Then, we calculate the ensemble average; we begin with first-order in both n_0 and m_0, and end with second-order in both. No closure assumptions are used. At first-order in n_0, we find precise agreement with the Foldy estimate for K, Eq. (1.3), correct to second-order in m_0. For the amplitude, A, we find agreement at first-order in m_0 with the Foldy estimate (given by Eq. (1.5)), but we find a discrepancy at second-order (see Eq. (5.25)).

Calculations at second-order in n_0 are more difficult because one has to introduce conditional probabilities, intended to prevent scatterers overlapping during the averaging process. We use a simple (but standard) pair-correlation function, giving what is known as a "hole correction", Eq. (5.13). This involves a new parameter, b, with $b \geqslant 2a$. Linton and Martin [6] found a formula for δ_2 in Eq. (1.2), making use of the Lax quasicrystalline approximation for their closure assumption (see Section 2.3). We compare the Linton–Martin formula (approximated to second-order in m_0) with our alternative approach: the agreement turns out to be perfect. One could view this agreement as supporting either of the two approaches described above, depending on one's point of view. We also obtain a new estimate for A, and we verify agreement with known results for "point scatterers" in the appropriate limit (see Section 5.3). Thus, our calculations show that good results may be obtained without closure assumptions, and so such methods deserve further investigation. In particular, extensions to three dimensions (random configurations of spheres) and to scatterers with a mass density that differs from that of the surrounding medium (including bubbles or rigid obstacles) should be made.

There is an extensive literature on multiple scattering by random configurations of identical obstacles. For thorough reviews, see, for example, [7] or [9].

2. Scattering by one cylinder: exact solution

Consider one circle of radius a, centred at the origin. We have $(\nabla^2 + k^2)u = 0$ for $r > a$ and $(\nabla^2 + k_0^2)u_0 = 0$ for $r < a$, where r and θ are plane polar coordinates and k and k_0 are real constants. The interface conditions are $u = u_0$ and $\partial u/\partial r = \partial u_0/\partial r$ on

$r = a$. Outside, we have $u = u_{\text{in}} + u_{\text{sc}}$ where $u_{\text{in}} = \mathrm{e}^{\mathrm{i}kx}$ and u_{sc} satisfies the Sommerfeld radiation condition. Throughout, we suppress a time dependence of $\mathrm{e}^{-\mathrm{i}\omega t}$.

The one-cylinder problem can be solved by separation of variables. We use the expansions [6]

$$u_{\text{in}}(r,\theta) = \sum_{n=-\infty}^{\infty} \mathrm{i}^n J_n(kr)\mathrm{e}^{\mathrm{i}n\theta}, \tag{2.1}$$

$$u_{\text{sc}}(r,\theta) = \sum_{n=-\infty}^{\infty} A_n Z_n H_n(kr)\mathrm{e}^{\mathrm{i}n\theta}, \tag{2.2}$$

$$u_0(r,\theta) = \sum_{n=-\infty}^{\infty} B_n J_n(k_0 r)\mathrm{e}^{\mathrm{i}n\theta}, \tag{2.3}$$

where J_n is a Bessel function, $H_n \equiv H_n^{(1)}$ is a Hankel function,

$$Z_n = [\mathrm{Re}\Delta_n]/\Delta_n = Z_{-n} \tag{2.4}$$

and

$$\Delta_n = H_n'(ka)J_n(k_0 a) - (k_0/k)J_n'(k_0 a)H_n(ka). \tag{2.5}$$

The interface conditions yield $A_n = -\mathrm{i}^n$ and

$$B_n = 2\mathrm{i}^{n+1}/(\pi ka\Delta_n). \tag{2.6}$$

The far-field pattern, $f(\theta)$, is defined by

$$u_{\text{sc}} \sim \sqrt{2/(\pi kr)}f(\theta)\exp(\mathrm{i}kr - \mathrm{i}\pi/4) \quad \text{as } r \to \infty. \tag{2.7}$$

Hence,

$$f(\theta) = -\sum_{n=-\infty}^{\infty} Z_n \mathrm{e}^{\mathrm{i}n\theta}. \tag{2.8}$$

2.1. Approximation for small m_0

For weak scattering, k_0 is close to k. To quantify this, we define the scattering strength, m_0, by

$$m_0 = 1 - (k_0/k)^2. \tag{2.9}$$

Then, for weak scattering, we approximate the exact solution, assuming that m_0 is small. From (2.9), we obtain

$$k_0/k \simeq 1 - \frac{1}{2}m_0 - \frac{1}{8}m_0^2 \tag{2.10}$$

and

$$F(k_0 a) \simeq F(ka) - \frac{1}{2}m_0 kaF'(ka) - \frac{1}{8}m_0^2 ka\{F'(ka) - kaF''(ka)\} \tag{2.11}$$

for any smooth function F. Then, some calculation gives

$$\Delta_n \simeq 2\mathrm{i}/(\pi ka) - (m_0/2)kad_n(ka) + (m_0^2/8)U_n, \tag{2.12}$$

where

$$d_n(x) = J_n'(x)H_n'(x) + [1 - (n/x)^2]J_n(x)H_n(x), \tag{2.13}$$

$$U_n = ka(kaJ_n'' - J_n')H_n' + (J_n' - ka[J_n'' + kaJ_n'''])H_n = 2ka[J_n(ka)H_n(ka) - d_n(ka) - \mathrm{i}/\pi] + 2\mathrm{i}n^2/(\pi ka) \tag{2.14}$$

and we have used

$$H_n'(x)J_n(x) - J_n'(x)H_n(x) = 2\mathrm{i}/(\pi x). \tag{2.15}$$

Then, from Eq. (2.12), the numerator in Eq. (2.4) is given approximately by

$$\mathrm{Re}\Delta_n \simeq -(m_0/2)ka\mathscr{J}_n(ka) + (m_0^2/8)S_n,$$

where

$$\mathscr{J}_n(ka) = \mathrm{Re}d_n(ka) = J_n^2(ka) - J_{n-1}(ka)J_{n+1}(ka), \tag{2.16}$$

$$S_n = \mathrm{Re}U_n = 2kaJ_{n-1}(ka)J_{n+1}(ka). \tag{2.17}$$

Hence, we obtain the following approximation for Z_n:

$$Z_n \simeq \frac{\pi}{4} m_0 \mathrm{i}(ka)^2 \mathscr{I}_n(ka) - \frac{\pi}{16} m_0^2 ka\{\mathrm{i}S_n - \pi(ka)^3 \mathscr{I}_n(ka) d_n(ka)\}. \tag{2.18}$$

Similarly, for the interior field, u_0, Eq. (2.6) gives

$$\mathrm{i}^{-n} B_n \simeq 1 - \frac{\mathrm{i}m_0}{4}\mu_n + \frac{m_0^2}{16}(\pi \mathrm{i}kaU_n - \mu_n^2) \quad \text{with} \quad \mu_n = \pi(ka)^2 d_n(ka).$$

Also, using Eq. (2.11),

$$J_n(k_0 r) \simeq J_n(kr) - (m_0/2)krJ_n'(kr) + (m_0^2/8)\{[n^2 - (kr)^2]J_n(kr) - 2krJ_n'(kr)\},$$

and then Eq. (2.3) gives

$$u_0(r,\theta) \simeq \mathrm{e}^{\mathrm{i}kx} - \frac{\mathrm{i}m_0}{4}\sum_{n=-\infty}^{\infty} \mathrm{i}^n u_n^{(1)} \mathrm{e}^{\mathrm{i}n\theta} + \frac{m_0^2}{16}\sum_{n=-\infty}^{\infty} \mathrm{i}^n u_n^{(2)} \mathrm{e}^{\mathrm{i}n\theta}, \tag{2.19}$$

where

$$u_n^{(1)} = \mu_n J_n(kr) - 2\mathrm{i}krJ_n'(kr),$$
$$u_n^{(2)} = \{2[n^2 - (kr)^2] + \pi \mathrm{i}kaU_n - \mu_n^2\}J_n(kr) + 2(\mathrm{i}\mu_n - 2)krJ_n'(kr).$$

2.2. Application to Foldy theory

Foldy's theory predicts that $\langle u \rangle = A\mathrm{e}^{\mathrm{i}Kx}$, where K is given by Eq. (1.3) and A is given by Eq. (1.5). Using Eqs. (2.8), (1.3) gives

$$K^2 - k^2 = 4\mathrm{i}n_0 \sum_{n=-\infty}^{\infty} Z_n \simeq -m_0 n_0 \pi(ka)^2 \sum_{n=-\infty}^{\infty} \mathscr{I}_n(ka) = -m_0 n_0 \pi(ka)^2 \tag{2.20}$$

to first-order in m_0. Here, we have used

$$\sum_{n=-\infty}^{\infty} \mathscr{I}_n = \sum_{n=-\infty}^{\infty} J_n^2 - \sum_{n=-\infty}^{\infty} J_n J_{n+2} = 1, \tag{2.21}$$

$$\sum_{n=-\infty}^{\infty} J_n^2(x) = 1, \tag{2.22}$$

$$\sum_{n=-\infty}^{\infty} J_n(x) J_{n+m}(x) = 0 \quad \text{for} \quad m = \pm 1, \pm 2, \ldots. \tag{2.23}$$

Also, from Eq. (1.5), we obtain $A \simeq 1 + \frac{1}{4}\pi m_0 n_0 a^2$.

For an approximation to second-order in m_0, use Eq. (2.18). The term in S_n does not contribute to $f(0)$: $\sum_{n=-\infty}^{\infty} S_n = 0$. To see this, use Eqs. (2.17) and (2.23). Thus, we obtain

$$f(0) = -\frac{1}{4}\mathrm{i}m_0\pi(ka)^2 + \frac{1}{4}\mathrm{i}m_0^2\pi^2(ka)^4 \mathscr{H}(ka), \tag{2.24}$$

where

$$\mathscr{H}(ka) = \frac{\mathrm{i}}{4}\sum_{n=-\infty}^{\infty} \mathscr{I}_n(ka) d_n(ka). \tag{2.25}$$

(Another formula for $\mathscr{H}(ka)$ is given in Appendix B.) Then Eq. (1.3) gives

$$K^2 - k^2 \simeq -m_0 n_0 \pi(ka)^2 + m_0^2 n_0 \pi^2 (ka)^4 \mathscr{H}(ka) \tag{2.26}$$

and Eq. (1.5) gives

$$A \simeq 1 + \frac{1}{4} m_0 n_0 \pi a^2 - \frac{1}{4} m_0^2 n_0 \pi^2 k^2 a^4 \mathscr{H}(ka). \tag{2.27}$$

2.3. Application to Linton–Martin theory

According to Linton and Martin [6, Eq. (80)], the second-order correction in Eq. (1.2) is given by

$$\delta_2 = 4\pi \mathrm{i}b^2 \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} Z_n Z_m d_{n-m}(kb), \tag{2.28}$$

where b is the parameter in the "hole correction", Eq. (5.13). (In the limit $kb \to 0$, δ_2 can be written as a certain integral of the far-field pattern; see [6, Eq. (86)].)

As a special case of Eq. (2.28), consider isotropic scattering, meaning that $Z_n = 0$ when $n \neq 0$, so that $f(\theta) = -Z_0$. Then, we obtain the estimate [6, Eq. (42)]

$$K^2 = k^2 - 4\mathrm{i}n_0 f(0) + 4\pi\mathrm{i}[n_0 b f(0)]^2 d_0(kb). \tag{2.29}$$

This is Eq. (5) in [10] when k_{eff} is replaced by k_0 in its right-hand side. (In fact, [10, Eq. (5)] is equivalent to [6, Eq. (40)].) From Eq. (2.18), the first-order approximation to Z_n is given by

$$Z_n \simeq (m_0/4)\pi\mathrm{i}(ka)^2 \mathscr{J}_n(ka), \tag{2.30}$$

whence Eq. (2.28) gives

$$\delta_2 \simeq m_0^2 \frac{\pi^3}{4\mathrm{i}} b^2 (ka)^4 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathscr{J}_n(ka)\mathscr{J}_m(ka) d_{n-m}(kb). \tag{2.31}$$

Thus, there are no contributions to K^2 that are proportional to $n_0^2 m_0$.

3. Lippmann–Schwinger equation

In two dimensions, the Lippmann–Schwinger integral equation is [11]

$$u(\mathbf{r}) = u_{\mathrm{in}}(\mathbf{r}) - k^2 \int G_0(\mathbf{r}, \mathbf{r}')m(\mathbf{r}')u(\mathbf{r}')\mathrm{d}V', \tag{3.1}$$

where u is the total field and

$$G_0(\mathbf{r}, \mathbf{r}') = (\mathrm{i}/4)H_0^{(1)}(k|\mathbf{r} - \mathbf{r}'|). \tag{3.2}$$

The governing partial differential equation is

$$\nabla^2 u + k^2 u = k^2 m(\mathbf{r})u, \tag{3.3}$$

with $m \equiv 0$ outside the scatterers. Thus, although the integration in Eq. (3.1) can be written as an integration over all space, it is really only over the scatterers.

It is known that Eq. (3.1) is always uniquely solvable. Moreover, the solution can be constructed by iterating the integral equation, under certain circumstances. We do not investigate these circumstances here. We simply accept the second iteration, giving

$$u(\mathbf{r}) \simeq u_{\mathrm{in}}(\mathbf{r}) - k^2 \int G_0(\mathbf{r}, \mathbf{r}')m(\mathbf{r}')u_{\mathrm{in}}(\mathbf{r}')\mathrm{d}V' + k^4 \int G_0(\mathbf{r}, \mathbf{r}')m(\mathbf{r}') \int G_0(\mathbf{r}', \mathbf{r}'')m(\mathbf{r}'')u_{\mathrm{in}}(\mathbf{r}'')\mathrm{d}V''\,\mathrm{d}V'. \tag{3.4}$$

We assume that the scatterers are D_i, centred at $\mathbf{r}_i$, $i = 1, 2, \ldots, N$. Each scatterer is a circular disc of radius a with constant strength m_0; when Eq. (2.9) is combined with Eq. (3.3), we see that $(\nabla^2 + k_0^2)u = 0$ holds inside the disc, as assumed in Section 2. Thus

$$(\nabla^2 + k^2)u = k^2 m_0 u(\mathbf{r}) \sum_{i=1}^{N} \chi_i(\mathbf{r}), \tag{3.5}$$

where χ_i is the characteristic function for D_i: $\chi_i(\mathbf{r}) = 1$ when $\mathbf{r} \in D_i$ and $\chi_i(\mathbf{r}) = 0$ when $\mathbf{r} \notin D_i$. The approximation (3.4) becomes

$$u(\mathbf{r}) \simeq u_{\mathrm{in}}(\mathbf{r}) - k^2 m_0 \sum_{i=1}^{N} \int_{D_i} G_0(\mathbf{r}, \mathbf{r}')u_{\mathrm{in}}(\mathbf{r}')\mathrm{d}V' + k^4 m_0^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{D_i} G_0(\mathbf{r}, \mathbf{r}') \int_{D_j} G_0(\mathbf{r}', \mathbf{r}'')u_{\mathrm{in}}(\mathbf{r}'')\mathrm{d}V''\,\mathrm{d}V'. \tag{3.6}$$

Using

$$(\nabla^2 + k^2) \int_B G_0(\mathbf{r}, \mathbf{r}')f(\mathbf{r}')\mathrm{d}V' = \begin{cases} 0, & \mathbf{r} \notin \overline{B}, \\ -f(\mathbf{r}), & \mathbf{r} \in B, \end{cases}$$

we see that Eq. (3.6) correctly generates a solution of $(\nabla^2 + k^2)u = 0$ outside the scatterers. On the other hand, if u_i denotes the field in D_i, Eq. (3.6) implies that

$$(\nabla^2 + k^2)u_i = k^2 m_0 u_{\mathrm{in}}(\mathbf{r}) - k^4 m_0^2 \sum_{j=1}^{N} \int_{D_j} G_0(\mathbf{r}, \mathbf{r}')u_{\mathrm{in}}(\mathbf{r}')\mathrm{d}V'$$

for $\mathbf{r} \in D_i$, whereas the exact solution satisfies $(\nabla^2 + k^2)u_i = k^2 m_0 u_i$.

We shall return to the formula (3.6) in Section 5, but first we consider scattering by one cylinder.

4. Scattering by one cylinder: iterative solution

Let us specialise the second-order solution (3.6) to a single cylinder. We have

$$u(\mathbf{r}) \simeq u_{\text{in}}(\mathbf{r}) + m_0 I_1(\mathbf{r}) + m_0^2 I_2(\mathbf{r}), \tag{4.1}$$

where

$$I_1(\mathbf{r}) = -k^2 \int_{D_0} G_0(\mathbf{r}, \mathbf{r}') u_{\text{in}}(\mathbf{r}') \mathrm{d}V', \tag{4.2}$$

$$I_2(\mathbf{r}) = -k^2 \int_{D_0} G_0(\mathbf{r}, \mathbf{r}') I_1(\mathbf{r}') \mathrm{d}V', \tag{4.3}$$

and D_0 is the circular disc of radius a, centred at the origin. In this section, we will evaluate I_1 and I_2. These quantities will be needed when we consider scattering by many circles (in Section 5). Also, as a check on our calculations, we verify that we recover the small-m_0 approximations to the exact solution given in Section 2.1.

4.1. Evaluation of I_1

With $\mathbf{r} = (r, \theta)$ and $\mathbf{r}' = (r', \theta')$, we use Eq. (2.1) in Eq. (4.2) to give

$$I_1(r, \theta) = -\sum_{n=-\infty}^{\infty} \mathrm{i}^n L_n(r, \theta), \tag{4.4}$$

where

$$L_n(r, \theta) = k^2 \int_0^a \int_0^{2\pi} G_0(\mathbf{r}, \mathbf{r}') J_n(kr') \mathrm{e}^{\mathrm{i}n\theta'} r' \mathrm{d}\theta' \, \mathrm{d}r'. \tag{4.5}$$

We give separate evaluations of $L_n(r, \theta)$ for $r > a$ and $r < a$.

4.1.1. Exterior field

For $r > a$, we use

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{\mathrm{i}}{4} \sum_{n=-\infty}^{\infty} H_n(kr) J_n(kr') \mathrm{e}^{\mathrm{i}n(\theta-\theta')}, \quad r > r', \tag{4.6}$$

in Eq. (4.5). This gives

$$L_n(r, \theta) = C_n^{(1)} H_n(kr) \mathrm{e}^{\mathrm{i}n\theta} \tag{4.7}$$

with

$$C_n^{(1)} = \frac{\pi \mathrm{i}}{2} k^2 \int_0^a J_n^2(kr') r' \mathrm{d}r' = \frac{\pi \mathrm{i}}{4} (ka)^2 \mathscr{J}_n(ka), \tag{4.8}$$

where $\mathscr{J}_n$ is defined by Eq. (2.16). Here, we have used [12], 7.14.1(10), p. 90,

$$\int w_n(kr) W_n(kr) r \, \mathrm{d}r = \frac{r^2}{4} \{2 w_n(kr) W_n(kr) - w_{n+1}(kr) W_{n-1}(kr) - w_{n-1}(kr) W_{n+1}(kr)\}, \tag{4.9}$$

where w_n and W_n are any two Bessel functions. In particular, we note that

$$\frac{2}{a^2} \int_0^a J_n^2(kr) r \, \mathrm{d}r = \mathscr{J}_n(ka), \tag{4.10}$$

a result that will be useful later.

The result (4.8) agrees with the exact solution (see Eq. (2.30)).

4.1.2. Interior field

For $r < a$, the calculation is more complicated. We obtain

$$L_n(r, \theta) = (\pi \mathrm{i}/8) \Lambda_n^{(1)}(kr) \mathrm{e}^{\mathrm{i}n\theta}, \quad r < a \tag{4.11}$$

with

$$\Lambda_n^{(1)}(kr) = 4k^2 H_n(kr) \int_0^r J_n^2(kr') r' \, \mathrm{d}r' + 4k^2 J_n(kr) \int_r^a H_n(kr') J_n(kr') r' \, \mathrm{d}r'. \tag{4.12}$$

Use of Eq. (4.9) gives

$$\Lambda_n^{(1)}(kr) = 2(kr)^2 H_n(kr)\{J_n^2(kr) - J_{n+1}(kr)J_{n-1}(kr)\} - (kr)^2 J_n(kr)\{2J_n(kr)H_n(kr) - J_{n+1}(kr)H_{n-1}(kr) - J_{n-1}(kr)H_{n+1}(kr)\} + (ka)^2 J_n(kr)\{2J_n(ka)H_n(ka) - J_{n+1}(ka)H_{n-1}(ka) - J_{n-1}(ka)H_{n+1}(ka)\}.$$

Suppressing the argument kr, the first line of this formula become

$$\begin{aligned}&(kr)^2[J_nJ_{n+1}H_{n-1} + J_nJ_{n-1}H_{n+1} - 2H_nJ_{n+1}J_{n-1}]\\ &\quad = J_n[nJ_n - krJ_n'][nH_n + krH_n'] + J_n[nJ_n + krJ_n'][nH_n - krH_n'] - 2H_n[nJ_n - krJ_n'][nJ_n + krJ_n']\\ &\quad = 2(kr)^2 J_n'[J_n'H_n - J_nH_n'] = -(4\mathrm{i}/\pi)krJ_n'(kr),\end{aligned}$$

whereas the second line reduces to $2(ka)^2 J_n(kr)d_n(ka)$ with d_n defined by Eq. (2.13). Hence,

$$\Lambda_n^{(1)}(kr) = -(4\mathrm{i}/\pi)krJ_n'(kr) + 2(ka)^2 J_n(kr)d_n(ka). \tag{4.13}$$

Combining Eqs. (4.1), (4.4), (4.11) and (4.13), we find agreement with the first-order approximation of the exact solution, Eq. (2.19).

4.2. Evaluation of I_2

As $r' < a$ in Eq. (4.3), we use Eq. (4.11) in Eq. (4.4) to give

$$I_2(\mathbf{r}) = \frac{\pi\mathrm{i}}{8}k^2 \sum_{m=-\infty}^{\infty} \mathrm{i}^m \int G_0(\mathbf{r},\mathbf{r}')\Lambda_m^{(1)}(kr')\mathrm{e}^{\mathrm{i}m\theta'}\,\mathrm{d}V'.$$

Again, the calculations depend on whether $r > a$ or $r < a$.

4.2.1. Exterior field

For $r > a$, we use Eq. (4.6). This gives

$$I_2(r,\theta) = -\sum_{n=-\infty}^{\infty} \mathrm{i}^n C_n^{(2)} H_n(kr)\mathrm{e}^{\mathrm{i}n\theta} \tag{4.14}$$

with

$$C_n^{(2)} = \frac{\pi^2 k^2}{16}\int_0^a J_n(kr')\Lambda_n^{(1)}(kr')r'\,\mathrm{d}r' \tag{4.15}$$

$$= (\pi ka/4)^2\{(ka)^2 d_n(ka)\mathscr{I}_n(ka) - (2\mathrm{i}/\pi)J_{n-1}(ka)J_{n+1}(ka)\}; \tag{4.16}$$

for the evaluation of the integral, see Eq. (A.7).

The result (4.16) agrees with the exact solution, see Eq. (2.18).

4.2.2. Interior field

For $r < a$, we obtain

$$I_2(r,\theta) = \left(\frac{\pi}{8\mathrm{i}}\right)^2 \sum_{n=-\infty}^{\infty} \mathrm{i}^n \Lambda_n^{(2)}(kr)\mathrm{e}^{\mathrm{i}n\theta}, \quad r < a \tag{4.17}$$

with

$$\Lambda_n^{(2)}(kr) = 4k^2 H_n(kr)\int_0^r J_n(kr')\Lambda_n^{(1)}(kr')r'\,\mathrm{d}r' + 4k^2 J_n(kr)\int_r^a H_n(kr')\Lambda_n^{(1)}(kr')r'\,\mathrm{d}r'.$$

Use of Eq. (4.13) gives

$$\Lambda_n^{(2)}(kr) = 2(ka)^2 d_n(ka)\Lambda_n^{(1)}(kr) - 8(\mathrm{i}/\pi)X_n \tag{4.18}$$

with

$$X_n = 2k^3 H_n(kr)\int_0^r J_n(ks)J_n'(ks)s^2\,\mathrm{d}s + 2k^3 J_n(kr)\int_r^a H_n(ks)J_n'(ks)s^2\,\mathrm{d}s.$$

To evaluate the remaining integrals, consider

$$\mathscr{I} = \int s^2 \frac{\mathrm{d}}{\mathrm{d}s}[J_n(ks)H_n(ks)]\mathrm{d}s.$$

Integrating by parts and then using Eq. (4.9) gives

$$\mathscr{I} = (s^2/2)(J_{n+1}H_{n-1} + J_{n-1}H_{n+1}),$$

suppressing the argument ks. Alternatively, we have

$$\mathscr{I} = k\int s^2(J_nH_n' + J_n'H_n)\mathrm{d}s = 2k\int s^2 J_n' H_n\,\mathrm{d}s + \frac{\mathrm{i}s^2}{\pi},$$

using the Wronskian, Eq. (2.15). Hence

$$2k\int s^2 J_n'(ks)H_n(ks)\mathrm{d}s = \frac{s^2}{2}\left\{J_{n+1}H_{n-1} + J_{n-1}H_{n+1} - \frac{2\mathrm{i}}{\pi}\right\} = s^2\{J_nH_n - d_n(ks) - \mathrm{i}/\pi\}.$$

In particular, the real part of this formula gives

$$2k\int s^2 J_n'(ks)J_n(ks)\mathrm{d}s = s^2 J_{n-1}(ks)J_{n+1}(ks). \tag{4.19}$$

Hence

$$X_n = X_n^{(1)} + X_n^{(2)}, \tag{4.20}$$

$$X_n^{(1)} = (kr)^2\{J_n(kr)[d_n(kr) + \mathrm{i}/\pi] - H_n(kr)\mathscr{J}_n(kr)\} = (\mathrm{i}kr/\pi)\{J_{n-1}(kr) - J_{n+1}(kr) + krJ_n(kr)\}, \tag{4.21}$$

$$X_n^{(2)} = (ka)^2 J_n(kr)[J_n(ka)H_n(ka) - d_n(ka) - \mathrm{i}/\pi] = (kaU_n/2 - n^2\mathrm{i}/\pi)J_n(kr). \tag{4.22}$$

U_n is defined by Eq. (2.14), $\Lambda_n^{(2)}$ is given by Eq. (4.18), and then I_2 is given by Eq. (4.17). When I_2 is substituted in Eq. (4.1), we find complete agreement with the second-order approximation to the exact solution, Eq. (2.19).

5. Scattering by *N* cylinders

Let us return to the second-order solution for scattering by N cylinders, given by Eq. (3.6). Introduce polar coordinates (r,θ) centred at the origin and (r_j,θ_j) centred at $\mathbf{r}_j = (x_j, y_j)$, the centre of the jth scatterer.

5.1. First-order in m_0

As in Section 4, using $u_{\text{in}} = \mathrm{e}^{\mathrm{i}kx} = \mathrm{e}^{\mathrm{i}kx_j}\mathrm{e}^{\mathrm{i}kr_j\cos\theta_j}$, we obtain

$$-k^2\int_{D_j} G_0(\mathbf{r},\mathbf{r}')u_{\text{in}}(\mathbf{r}')\mathrm{d}V' = \mathrm{e}^{\mathrm{i}kx_j}I_1(r_j,\theta_j) \tag{5.1}$$

with $r_j \equiv |\mathbf{r} - \mathbf{r}_j|$, so that, to first-order in m_0, Eq. (3.6) gives

$$u(\mathbf{r}) \simeq \mathrm{e}^{\mathrm{i}kx} + m_0\sum_{j=1}^{N}\mathrm{e}^{\mathrm{i}kx_j}I_1(r_j,\theta_j). \tag{5.2}$$

Next, we calculate the ensemble average of u, $\langle u\rangle$. At this stage, we can assume that the scatterers are independent (uncorrelated); they are also indistinguishable. The result is

$$\langle u(\mathbf{r})\rangle = \mathrm{e}^{\mathrm{i}kx} + m_0n_0\int_{B_N}\mathrm{e}^{\mathrm{i}kx_1}I_1(r_1,\theta_1)\mathrm{d}x_1\,\mathrm{d}y_1, \tag{5.3}$$

where B_N is the region occupied by the N circles and n_0 is the number of circles per unit area, so that B_N has area N/n_0.

In order to do explicit calculations, we now let $N\to\infty$. One option would be for B_N to become a slab, $0<x<L$, say. We prefer to let B_N become the half-plane $x>0$, but then we have to take care with the convergence of various integrals as $x\to\infty$: we replace $\mathrm{e}^{\mathrm{i}kx_1}$ by $\mathrm{e}^{\mathrm{i}\kappa x_1}$, with $\mathrm{Im}\,\kappa>0$, and then let $\kappa\to k$ at the end of the calculation.

Thus, in the limit $N\to\infty$, the integral in Eq. (5.3) consists of an integral over the circular disc $r_1<a$ ($r_1^2 = (x_1-x)^2 + y_1^2$) plus an integral over the remainder of the half-plane $x_1>0$; we denote this region by disc′. We can assume here that $x>a$ so that the disc does not meet the line $x_1=0$.

Exact calculation gives (see Appendix A)

$$\int_{r_1<a}\mathrm{e}^{\mathrm{i}kx_1}I_1(r_1,\theta_1)\mathrm{d}x_1\,\mathrm{d}y_1 = -\pi^2a^2\mathrm{e}^{\mathrm{i}kx}(ka)^2\mathscr{H}(ka), \tag{5.4}$$

$$\lim_{\kappa\to k}\int_{\text{disc}'}\mathrm{e}^{\mathrm{i}\kappa x_1}I_1(r_1,\theta_1)\mathrm{d}x_1\,\mathrm{d}y_1 = \frac{\pi a^2}{4}\mathrm{e}^{\mathrm{i}kx}\{1 - 2\mathrm{i}kx + 4\pi(ka)^2\mathscr{H}(ka)\}, \tag{5.5}$$

where $\mathscr{H}$ is defined by Eq. (2.25). When these two expressions are added together, the terms in $\mathscr{H}$ cancel, and then Eq. (5.3) gives

$$\langle u(\mathbf{r})\rangle = \mathrm{e}^{\mathrm{i}kx}\{1 + m_0n_0(\pi/4)a^2(1 - 2\mathrm{i}kx)\}, \quad x>a. \tag{5.6}$$

This should be compared with Eq. (1.4), $\langle u(\mathbf{r})\rangle = A\mathrm{e}^{\mathrm{i}Kx}$. Write

$$A \simeq 1 + m_0A_1 + m_0^2A_2 \quad \text{and} \quad K \simeq k + m_0k_1 + m_0^2k_2,$$

so that $K^2 \simeq k^2 + 2m_0kk_1 + m_0^2(k_1^2 + 2kk_2)$. Then,

$$Ae^{iKx} = e^{ikx}\{1 + m_0(A_1 + ik_1x) + m_0^2(A_2 + ix[A_1k_1 + k_2] - k_1^2x^2/2)\}; \tag{5.7}$$

the terms involving m_0^2 will be used later. Comparing Eqs. (5.6) and (5.7) gives $A_1 = \phi/4$ and $k_1 = -k\phi/2$, where we have defined the area fraction occupied by the scatterers, ϕ, by

$$\phi = n_0\pi a^2.$$

Hence, $A \simeq 1 + m_0\phi/4$ and $K^2 \simeq k^2(1 - m_0\phi)$, in agreement with the Foldy estimates, Eqs. (2.20) and (2.27), correct to first-order in m_0.

The idea of rewriting an expansion such as Eq. (5.6) in the form Eq. (1.4) is well established (see [4], §7.4.2, for example).

5.2. Second-order in m_0

At second-order in m_0, we add the last term in Eq. (3.6) to the right-hand side of Eq. (5.2). Using Eq. (5.1), this term becomes

$$-k^2m_0^2\sum_{i=1}^{N}\sum_{j=1}^{N}e^{ikx_j}\int_{D_i}G_0(\mathbf{r},\mathbf{r}')I_1(r_j',\theta_j')\mathrm{d}V' = u_2(\mathbf{r}), \tag{5.8}$$

say; recall that r_j, θ_j are polar coordinates centred at (x_j, y_j), the centre of the jth disc, D_j.

Evidently, the evaluation of the integrals in Eq. (5.8) will be different if $i = j$ or $i \neq j$, and so we write

$$u_2 = u_2^{(1)} + u_2^{(2)}, \tag{5.9}$$

where

$$u_2^{(1)}(\mathbf{r}) = -k^2m_0^2\sum_{j=1}^{N}e^{ikx_j}\int_{D_j}G_0(\mathbf{r},\mathbf{r}')I_1(r_j',\theta_j')\mathrm{d}V' \tag{5.10}$$

and $u_2^{(2)} = u_2 - u_2^{(1)}$.

5.2.1. Calculation of $\langle u_2^{(1)}\rangle$

From Eq. (4.3), we obtain

$$u_2^{(1)}(\mathbf{r}) = m_0^2\sum_{j=1}^{N}e^{ikx_j}I_2(r_j,\theta_j),$$

where I_2 is given by Eq. (4.14) for $r_j > a$ and by Eq. (4.17) for $r_j < a$. Then, proceeding as in Section 5.1, we obtain

$$\langle u_2^{(1)}(\mathbf{r})\rangle = e^{ikx}m_0^2(\phi/4)\{P_0(ka) + (2ikx - 1)Q_0(ka)\}, \quad x > a, \tag{5.11}$$

where $P_0(ka)$ is given by Eq. (A.9) and $Q_0(ka) = \pi(ka)^2\mathscr{H}(ka)$ (see Appendix A for details of the calculation).

5.2.2. Calculation of $\langle u_2^{(2)}\rangle$

When $i \neq j, r_j' > a$ (as the scatterers are not allowed to overlap) and so we can use Eq. (4.7) in Eq. (4.4):

$$\int_{D_i}G_0(\mathbf{r},\mathbf{r}')I_1(r_j',\theta_j')\mathrm{d}V' = -\sum_{n=-\infty}^{\infty}i^nC_n^{(1)}\int_{D_i}G_0(\mathbf{r},\mathbf{r}')H_n(kr_j')e^{in\theta_j'}\,\mathrm{d}V'.$$

Here, r_j', θ_j' are the polar coordinates of the point at $\mathbf{r}'$ with respect to the centre of D_j. So, to integrate over D_i, we need Graf's addition theorem to express $H_n(kr_j')e^{in\theta_j'}$ in terms of $J_m(kr_i')e^{im\theta_i'}$:

$$H_n(kr_j')e^{in\theta_j'} = \sum_{m=-\infty}^{\infty}J_m(kr_i')H_{n-m}(kR_{ji})e^{im\theta_i'}e^{i(n-m)\alpha_{ji}},$$

where $x_i - x_j = R_{ji}\cos\alpha_{ji}$ and $y_i - y_j = R_{ji}\sin\alpha_{ji}$ (see [6], Fig. 1 for a diagram showing notation). Hence,

$$-k^2\int_{D_i}G_0(\mathbf{r},\mathbf{r}')I_1(r_j',\theta_j')\mathrm{d}V' = \sum_{m,n}i^nC_n^{(1)}H_{n-m}(kR_{ji})e^{i(n-m)\alpha_{ji}}L_m(r_i,\theta_i),$$

where L_n is defined by Eq. (4.5) and we have used the short-hand notation

$$\sum_{m,n} \equiv \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}.$$

Thus, from Eqs. (5.8)–(5.10), we obtain

$$u_2^{(2)}(\mathbf{r}) = m_0^2\sum_{m,n}C_n^{(1)}\Omega_{mn}(\mathbf{r}) \tag{5.12}$$

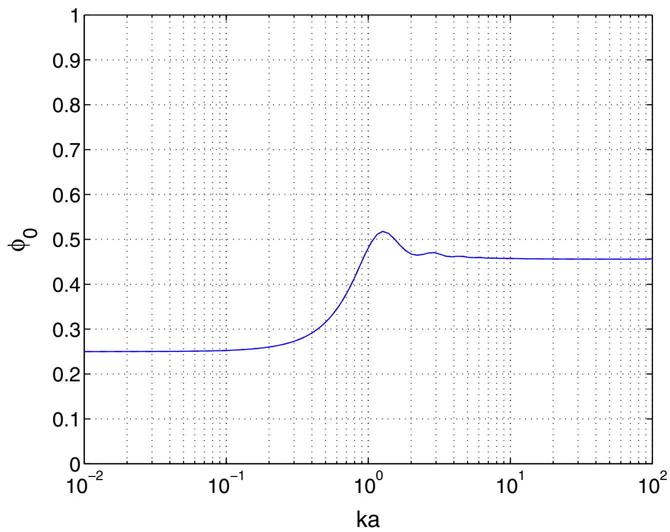

Fig. 1. The function $\phi_0(ka)$: when $b = 2a$, the imaginary part of the effective wavenumber, K_i, is positive for $0 \leqslant \phi < \phi_0(ka)$.

with

$$\Omega_{mn} = \mathrm{i}^n \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \mathrm{e}^{\mathrm{i}kx_j} H_{n-m}(kR_{ji}) \mathrm{e}^{\mathrm{i}(n-m)\alpha_{ji}} L_m(r_i, \theta_i).$$

To compute the ensemble average of Ω_{mn}, we need a conditional probability. We use $p(\mathbf{r}_i) = n_0/N$ (as above) and

$$p(\mathbf{r}_j|\mathbf{r}_i) = (n_0/N)H(R_{ij} - b), \tag{5.13}$$

where H is the Heaviside unit function and b is the "hole radius"; we take $b \geqslant 2a$ to ensure that the circular scatterers do not overlap during the averaging. Then, as the scatterers are indistinguishable, we obtain

$$\langle \Omega_{mn} \rangle = \mathrm{i}^n n_0^2 \frac{N-1}{N} \int_{B_N} L_m(r_1, \theta_1) \int_{B_N} H(R_{21} - b) \mathrm{e}^{\mathrm{i}kx_2} H_{n-m}(kR_{21}) \mathrm{e}^{\mathrm{i}(n-m)\alpha_{21}} \mathrm{d}V_2 \, \mathrm{d}V_1.$$

As before, we replace $\mathrm{e}^{\mathrm{i}kx_2}$ by $\mathrm{e}^{\mathrm{i}\kappa x_2}$ with $\mathrm{Im}\,\kappa > 0$, we let $N \to \infty$ so that B_N becomes the half-plane $x > 0$, we evaluate the integrals and then we let $\kappa \to k$. To begin, we note that the inner integral is very similar to Eq. (A.1); we take its value as

$$\frac{2}{\mathrm{i}k} (-\mathrm{i})^{n-m} \frac{(\kappa + k)\mathrm{e}^{\mathrm{i}kx_1} + \pi \mathrm{i}k \mathrm{e}^{\mathrm{i}\kappa x_1} \mathcal{N}_{n-m}(\kappa b)}{k^2 - \kappa^2} \tag{5.14}$$

with $\mathcal{N}_n$ defined by Eq. (A.3). This result is exact when $x_1 > b$ but it is an approximation when $0 < x_1 < b$ (see the discussion below Eq. (A.2)). Using Eq. (5.14) for all $x_1 > 0$, we obtain

$$\langle \Omega_{mn} \rangle = \frac{2n_0^2}{\mathrm{i}k(k^2 - \kappa^2)} \{(\kappa + k)\mathcal{L}_m(k) + \pi \mathrm{i}k \mathcal{N}_{n-m}(\kappa b)\mathcal{L}_m(\kappa)\},$$

where

$$\mathcal{L}_m(\kappa) = \mathrm{i}^m \int_{x_1>0} \mathrm{e}^{\mathrm{i}\kappa x_1} L_m(r_1, \theta_1) \mathrm{d}x_1 \mathrm{d}y_1 = \frac{2C_m^{(1)}}{k^2 - \kappa^2} \left\{ \frac{\kappa + k}{\mathrm{i}k} \mathrm{e}^{\mathrm{i}kx} + \pi \mathrm{e}^{\mathrm{i}\kappa x} \mathcal{N}_m(\kappa a) \right\} + \frac{\pi^2}{4} \mathrm{i} \mathrm{e}^{\mathrm{i}\kappa x} \int_0^a J_m(\kappa r_1) \Lambda_m^{(1)}(kr_1) r_1 \, \mathrm{d}r_1. \tag{5.15}$$

Here, we have used Eqs. (4.7) and (4.11), and made calculations similar to Eqs. (A.2) and (A.6). Letting $\kappa \to k$, we obtain

$$\lim_{\kappa \to k} \langle \Omega_{mn} \rangle = n_0^2 k^{-2} \mathrm{i} \{ [1 + \pi \mathrm{i}(kb)^2 d_{n-m}(kb)] \mathcal{L}_m(k) - 2k \mathcal{L}'_m(k) \}, \tag{5.16}$$

where we have used $\mathcal{N}_m(kb) = 2\mathrm{i}/\pi$ and $\mathcal{N}'_m(kb) = kbd_m(kb)$. Letting $\kappa \to k$ in Eq. (5.15) gives

$$\mathcal{L}_m(k) = k^{-2} \mathrm{e}^{\mathrm{i}kx} \{ p_m^{(0)} + (2\mathrm{i}kx - 1) p_m^{(1)} \}, \tag{5.17}$$

where

$$p_m^{(0)} = \frac{1}{2} \pi (ka)^2 J_{m-1}(ka) J_{m+1}(ka), \quad p_m^{(1)} = -\mathrm{i}C_m^{(1)}$$

and we have used Eq. (A.7). Differentiating Eq. (5.15) with respect to κ gives

$$\mathscr{L}'_m(\kappa) = \frac{4\kappa C_m^{(1)}}{(k^2-\kappa^2)^2}\left\{\frac{\kappa+k}{\mathrm{i}k}\mathrm{e}^{\mathrm{i}kx} + \pi\mathrm{e}^{\mathrm{i}\kappa x}\mathscr{N}_m(\kappa a)\right\} + \frac{2C_m^{(1)}}{k^2-\kappa^2}\left\{\frac{\mathrm{e}^{\mathrm{i}kx}}{\mathrm{i}k} + \pi\mathrm{e}^{\mathrm{i}\kappa x}[\mathrm{i}x\mathscr{N}_m(\kappa a) + a\mathscr{N}'_m(\kappa a)]\right\} + \frac{\pi^2}{4}\mathrm{i}\mathrm{e}^{\mathrm{i}\kappa x}\int_0^a[\mathrm{i}xJ_m(\kappa r_1) + r_1J'_m(\kappa r_1)]\Lambda_m^{(1)}(kr_1)r_1\,\mathrm{d}r_1.$$

In the limit $\kappa \to k$, we find (after some calculation)

$$2k\mathscr{L}'_m(k) = k^{-2}\mathrm{e}^{\mathrm{i}kx}\{q_m^{(0)} + (2\mathrm{i}kx-1)q_m^{(1)} + (kx)^2q_m^{(2)}\}, \tag{5.18}$$

where $q_m^{(1)} = \mathrm{i}C_m^{(1)} + p_m^{(0)}, q_m^{(2)} = 2\mathrm{i}C_m^{(1)}$ and

$$q_m^{(0)} = \pi(ka)^2C_m^{(1)}[d_m(ka) - \mathscr{N}''_m(ka)] + p_m^{(0)} + \frac{\mathrm{i}\pi^2}{2}k^3\int_0^a J'_m(kr)\Lambda_m^{(1)}(kr)r^2\,\mathrm{d}r;$$

here, we have used Eqs. (4.8) and (A.7). Simplification gives

$$q_m^{(0)} = 2\pi(ka)^2C_m^{(1)}\{d_m(ka) - J_m(ka)H_m(ka) + (\mathrm{i}/\pi)[1 - m^2/(ka)^2]\} + [1 + \pi\mathrm{i}(ka)^2d_m(ka)]p_m^{(0)} + 2\pi\int_0^{ka}[J'_m(x)]^2x^3\,\mathrm{d}x. \tag{5.19}$$

Substituting Eqs. (5.17) and (5.18) in Eq. (5.16) gives

$$\lim_{\kappa\to k}\langle\Omega_{mn}\rangle = \mathrm{i}n_0^2k^{-4}\mathrm{e}^{\mathrm{i}kx}\{P_{mn}^{(0)} + (2\mathrm{i}kx-1)P_{mn}^{(1)} - (kx)^2q_m^{(2)}\},$$

where $P_{mn}^{(j)} = [1 + \pi\mathrm{i}(kb)^2d_{n-m}(kb)]p_m^{(j)} - q_m^{(j)},\ j = 0, 1.$

Next, we sum over m. From Eqs. (2.21) and (4.8), we have

$$\sum_{m=-\infty}^{\infty} C_m^{(1)} = \frac{\pi\mathrm{i}}{4}(ka)^2. \tag{5.20}$$

Hence, using Eq. (2.23), we obtain

$$\sum_{m=-\infty}^{\infty} P_{mn}^{(1)} = \frac{\pi}{2}(ka)^2 + \pi(kb)^2\sum_{m=-\infty}^{\infty} C_m^{(1)}d_{n-m}(kb).$$

Then, from Eq. (5.12), we find that

$$\langle u_2^{(2)}(\mathbf{r})\rangle = \mathrm{e}^{\mathrm{i}kx}m_0^2(\phi^2/4)\{P_1 + (2\mathrm{i}kx-1)Q_1 + (kx)^2R_1\}, \tag{5.21}$$

where

$$P_1 = \frac{4\mathrm{i}}{\pi^2(ka)^4}\sum_{m,n}C_n^{(1)}P_{mn}^{(0)} = \frac{(kb)^2}{\mathrm{i}(ka)^2}\sum_{m,n}p_m^{(0)}\mathscr{J}_n(ka)d_{n-m}(kb) + \frac{1}{\pi(ka)^2}\sum_{m=-\infty}^{\infty}q_m^{(0)}, \tag{5.22}$$

$$Q_1 = \frac{4\mathrm{i}}{\pi^2(ka)^4}\sum_{m,n}C_n^{(1)}P_{mn}^{(1)} = -\frac{1}{2} + \frac{4\mathrm{i}}{\pi}\frac{(kb)^2}{(ka)^4}\sum_{m,n}C_m^{(1)}C_n^{(1)}d_{n-m}(kb), \tag{5.23}$$

$$R_1 = -\frac{4\mathrm{i}}{\pi^2(ka)^4}\sum_{m,n}C_n^{(1)}q_m^{(2)} = -\frac{1}{2}.$$

We add Eqs. (5.11) and (5.21) to the right-hand side of Eq. (5.6), giving the approximation

$$\langle u(\mathbf{r})\rangle = \mathrm{e}^{\mathrm{i}kx}\{1 + m_0(\phi/4)(1 + m_0[P_0 - Q_0] + m_0\phi[P_1 - Q_1]) - \mathrm{i}kxm_0(\phi/2)(1 - m_0Q_0 - m_0\phi Q_1) - (kx)^2m_0^2\phi^2/8\}. \tag{5.24}$$

When this is compared with Eq. (5.7), we find that $A_1 = \phi/4$ and $k_1 = -k\phi/2$, as before, the terms in x^2 agree,

$$A_2 = (P_0 - Q_0 + \phi[P_1 - Q_1])\phi/4 \quad \text{and} \quad k_2/k = Q_0\phi/2 + (4Q_1 + 1)\phi^2/8.$$

Hence, we obtain the approximation

$$K^2/k^2 \simeq 1 - m_0\phi + m_0^2\phi Q_0 + m_0^2\phi^2(Q_1 + 1/2).$$

The last term implies that δ_2 in Eq. (1.2) is given by

$$\delta_2 \simeq k^2m_0^2(\pi a^2)^2(Q_1 + 1/2) = 4\pi\mathrm{i}m_0^2b^2\sum_{m,n}C_m^{(1)}C_n^{(1)}d_{n-m}(kb),$$

where we have used Eq. (5.23). Then, using Eq. (4.8), we find precise agreement with Eq. (2.31). We also find the approximation

$$A \simeq 1 + m_0\phi/4 + m_0^2(P_0 - Q_0)\phi/4 + m_0^2(P_1 - Q_1)\phi^2/4. \tag{5.25}$$

The term involving $m_0^2\phi$ in Eq. (5.25) differs from the Foldy approximation, Eq. (2.27), by the presence of P_0. This quantity came from an exact calculation of $\langle u_2^{(1)}\rangle$ (see Eq. (5.11)). Specifically, P_0 came from a certain integral over the interior of a typical scatterer (see Eq. (A.8)). We have checked that P_0 does not vanish identically (see Eq. (A.10)).

5.3. Comparison with "point scatterers"

Similar results have been obtained by Maurel [5] for uncorrelated configurations of "point scatterers". For the deterministic problem, the governing partial differential equation is

$$(\nabla^2 + k^2)u = k^2 m_0 \pi a^2 u(\mathbf{r}) \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i),$$

where δ is the Dirac delta function. The factor πa^2 ensures agreement with the right-hand side of Eq. (3.5): $\int u\chi_i \, dV \simeq \pi a^2 u(\mathbf{r}_i)$ when D_i is small. Maurel [5] found the approximation

$$\langle u(\mathbf{r})\rangle = e^{ikx}\{1 + m_0(\phi/4)[1 - m_0\mathcal{Q} + m_0\phi/2] - ikxm_0(\phi/2)[1 - m_0\mathcal{Q} + m_0\phi/2] - (kx)^2 m_0^2\phi^2/8\}, \tag{5.26}$$

where $\mathcal{Q}(ka) = \pi(i/4)(ka)^2 H_0(ka)$.

From (5.26), the estimates

$$A \simeq 1 + m_0\phi/4 - m_0^2(\phi/4)\mathcal{Q} + m_0^2\phi^2/8,$$
$$K/k \simeq 1 - m_0\phi/2 + m_0^2(\phi/2)\mathcal{Q} - m_0^2\phi^2/8$$

follow; squaring the last formula gives $K^2/k^2 \simeq 1 - m_0\phi + m_0^2\phi\mathcal{Q}$, with no term in $m_0^2\phi^2$.

Intuitively, the point-scatterer limit should correspond to $ka \to 0$. If we compare Eqs. (5.24) and (5.26), we see that we will have agreement if

$$P_0 - Q_0 \sim -\mathcal{Q}, \quad P_1 - Q_1 \sim \frac{1}{2}, \quad Q_0 \sim \mathcal{Q} \quad \text{and} \quad Q_1 \sim -\frac{1}{2}. \tag{5.27}$$

The third of these follows quickly from the definition of Q_0, whereas the first follows from the fact that P_0 is smaller than Q_0 as $ka \to 0$ (see Eq. (A.11)). The second and fourth of (5.27) involve P_1 and Q_1, and these depend on kb as well as ka. From Eq. (5.23), we find that

$$Q_1 + 1/2 \sim -\pi(i/4)(kb)^2 d_0(kb) \quad \text{as} \quad ka \to 0;$$

the limiting value vanishes when $kb \to 0$, which is the appropriate limit for uncorrelated scatterers. Finally, consider P_1, defined by Eq. (5.22) as a double-sum term plus a single-sum term. In the double sum, the dominant contributions come from $\mathcal{J}_0 \sim 1$, $p_0^{(0)} \sim (\pi/8)(ka)^4$ and $p_{\pm1}^{(0)} \sim p_0^{(0)}/2$ as $ka \to 0$. Hence, asymptotically, the double-sum term is

$$-i(\pi/8)(ka)^2(kb)^2[d_0(kb) + d_1(kb)] \to 0 \quad \text{as} \quad ka \to 0.$$

Making use of Eqs. (5.19), (5.20),

$$\sum_{m=-\infty}^{\infty} p_m^{(0)} = 0 \quad \text{and} \quad \sum_{m=-\infty}^{\infty} [J_m'(x)]^2 = \frac{1}{2},$$

the single-sum term in Eq. (5.22),

$$\frac{1}{\pi(ka)^2}\sum_{m=-\infty}^{\infty} q_m^{(0)} = -\frac{1}{4}(ka)^2 + \sum_{m=-\infty}^{\infty}\{\mathcal{J}_m(ka)\gamma_m(ka) + i d_m(ka)p_m^{(0)}\}, \tag{5.28}$$

exactly, where $\gamma_m(x) = (i\pi/2)x^2[d_m(x) - J_m(x)H_m(x)] + m^2/2$. We have $\gamma_0(x) \sim x^2/2, d_0(x) \sim H_0(x), \gamma_m(x) \sim |m| + m^2/2$ and $x^2 d_m(x) \sim 2i|m|/\pi$ for $m \neq 0$, so that the largest terms in Eq. (5.28) are $O((ka)^2)$ as $ka \to 0$. Thus, all of Eq. (5.27) hold in the limit $ka \to 0$, provided $kb \to 0$ too. It follows that we recover the point-scatterer results if we allow the scatterers to shrink ($ka \to 0$) and to become uncorrelated.

6. Numerical results and summary

In this section, we give some quantitative illustrations of the analytical results above. However, let us first summarise the various theories. We distinguish between Foldy-type theories and iterative theories.

6.1. Foldy-type theories

The simplest is "basic" Foldy theory: it predicts that $K^2 = k^2 - 4in_0 f(0)$ and that A is given by Eq. (1.5); it is first order in n_0; it assumes that the scatterers are statistically independent (possible overlaps are ignored at first-order); it uses the forward-scattered far-field pattern for one scatterer, $f(0)$, calculated (exactly) taking proper account of the interior wavefield and density differences, if present; and it makes essential use of the Foldy closure assumption.

As a special case, we have "weak Foldy". This occurs when the scattering from each individual scatterer is weak, meaning that each scatterer is penetrable with a sound-speed that is close to that in the exterior; this closeness is measured by the small parameter, m_0. (For simplicity, we do not permit density differences here.) The "weak-Foldy" results are given in Section 2.2. Thus, K^2 is given by Eq. (2.26). Also,

$$|A| = 1 + m_0\phi/4 - m_0^2(\phi/4)\mathrm{Re}Q_0 \tag{6.1}$$

(when $u_{\mathrm{in}} = \mathrm{e}^{\mathrm{i}kx}$) and $K_\mathrm{i}/k = m_0^2(\phi/2)\mathrm{Im}\,Q_0$, where $K_\mathrm{i} = \mathrm{Im}\,K$, $\phi = n_0\pi a^2$ is the filling fraction or area fraction occupied by the scatterers, $Q_0(ka) = \pi(ka)^2\mathscr{H}$ and $\mathscr{H}(ka)$ is given by Eq. (2.25) or Eq. (B.1). Note that

$$\frac{K_\mathrm{i}}{k} = m_0^2\phi(ka)^2\frac{\pi}{8}\sum_{n=-\infty}^{\infty}\mathscr{J}_n^2(ka), \tag{6.2}$$

which is positive, implying attenuation with x.

Going beyond Foldy, we can seek corrections proportional to n_0^2. This is more difficult because pair-correlations must be used in order to prevent finite-sized scatterers from overlapping during averaging. The "Linton–Martin" correction to basic Foldy for K is $n_0^2\delta_2$ with δ_2 given by Eq. (2.28). It was derived using the Lax quasicrystalline approximation as closure assumption. Again, as with basic Foldy, there is a weak version; it gives a term in $n_0^2m_0^2$, see Section 2.3, especially Eq. (2.31). The Linton–Martin theory does not give any estimate for A.

Note that Foldy and Linton–Martin theories do not assume explicitly that ka is small. However, letting $ka \to 0$ gives results for very small scatterers, and these can be compared with results for so-called "point scatterers"; this was done in Section 5.3. In particular, the point-scatterer limit gives

$$|A| = 1 + m_0\phi/4 - m_0^2(\phi/4)\mathrm{Re}\mathscr{Q} + m_0^2\phi^2/8 \tag{6.3}$$

and $K_\mathrm{i}/k = m_0^2(\phi/2)\mathrm{Im}\,\mathscr{Q}$, where $\mathscr{Q} = \pi(\mathrm{i}/4)(ka)^2H_0(ka)$ so that

$$\mathrm{Re}\mathscr{Q} = -(\pi/4)(ka)^2Y_0(ka) \sim -(1/2)(ka)^2\log ka, \tag{6.4}$$

$$\mathrm{Im}\,\mathscr{Q} = (\pi/4)(ka)^2J_0(ka) \sim (\pi/4)(ka)^2. \tag{6.5}$$

6.2. Iterative theories

In this paper, we began with the Lippmann–Schwinger equation, which we solved by iteration for weak scattering. At both first and second-order in m_0, we obtained *exactly* the same expressions for K as those obtained by "weak Foldy" and "weak Linton–Martin". No closure assumptions were used. However, there is one difference in the result for A:

$$|A| = 1 + m_0\phi/4 - m_0^2(\phi/4)\mathrm{Re}(Q_0 - P_0) + m_0^2(\phi^2/4)\mathrm{Re}(P_1 - Q_1). \tag{6.6}$$

Thus, the term P_0 is absent from Eq. (6.1). Also,

$$\frac{K_\mathrm{i}}{k} = m_0^2\phi(ka)^2\frac{\pi}{8}\left\{\sum_{n=-\infty}^{\infty}\mathscr{J}_n^2(ka) - \phi(b/a)^2\sum_{m,n}\mathscr{J}_m(ka)\mathscr{J}_n(ka)\mathscr{J}_{n-m}(kb)\right\}; \tag{6.7}$$

this expression can become negative (see Section 6.3). Eqs. (6.6) and (6.7) do reduce to the point-scatterer limits when both ka and $kb \to 0$.

6.3. Numerical results

The various theories described above give numerical predictions that are close, at least for small values of ϕ and ka. Rather than plot curves that are almost superimposed, we give a few numerical values. In Table 1, we give values of $|A|$ at $ka = 1$ for three values of ϕ and three values of m_0. In each case, the top value is the Foldy prediction (Eq. (6.1)), the middle value is the iterative point-scatterer result (Eq. (6.3) with the approximation (6.4)) and the bottom value is the iterative finite-size result (Eq. (6.6) with hole radius $b = 2a$). Corresponding results for $\mathrm{Im}\,K/k$ are given in Table 2: the Foldy prediction is Eq. (6.2), the

Table 1
Computed values of $|A|$ at $ka = 1$ for three values of ϕ and three values of m_0

	$m_0 = 0.1$	$m_0 = 0.5$	$m_0 = 0.9$
$\phi = 0.1$	1.0025	1.0118	1.0202
	1.0025	1.0128	1.0235
	1.0025	1.0126	1.0227
$\phi = 0.2$	1.0049	1.0236	1.0404
	1.0051	1.0262	1.0491
	1.0050	1.0260	1.0481
$\phi = 0.4$	1.0099	1.0472	1.0808
	1.0102	1.0550	1.1062
	1.0102	1.0553	1.1071

In each case, the top value is the Foldy prediction, the middle value is the iterative point-scatterer result, and the bottom value is the iterative finite-size result (for which $b = 2a$).

Table 2
Computed values of $\mathrm{Im}\,K/k$ at $ka = 1$ for three values of ϕ and three values of m_0

	$m_0 = 0.1$	$m_0 = 0.5$	$m_0 = 0.9$
$\phi = 0.1$	2.4721×10^{-4}	6.1802×10^{-3}	2.0024×10^{-2}
	3.9270×10^{-4}	9.8175×10^{-3}	3.1809×10^{-2}
	1.9587×10^{-4}	4.8966×10^{-3}	1.5865×10^{-2}
$\phi = 0.2$	4.9441×10^{-4}	12.3603×10^{-3}	4.0047×10^{-2}
	7.8540×10^{-4}	19.6350×10^{-3}	6.3617×10^{-2}
	2.8905×10^{-4}	7.2262×10^{-3}	2.3413×10^{-2}
$\phi = 0.4$	9.8882×10^{-4}	24.7206×10^{-3}	8.0095×10^{-2}
	15.7080×10^{-4}	39.2699×10^{-3}	12.7235×10^{-2}
	1.6737×10^{-4}	4.1843×10^{-3}	1.3557×10^{-2}

In each case, the top value is the Foldy prediction, the middle value is the iterative point-scatterer result, and the bottom value is the iterative finite-size result (for which $b = 2a$).

iterative point-scatterer result is given just below Eq. (6.3) and the iterative finite-size result is Eq. (6.7), again with $b = 2a$. We notice that the last of these, which is presumably more accurate, predicts weaker attenuation, especially as ϕ and m_0 get larger.

The iterative finite-size prediction for $\mathrm{Im}\,K$, Eq. (6.7), fails for values of ϕ that are sufficiently large. To see this, write Eq. (6.7) as

$$K_{\mathrm{i}}/k = m_0^2\phi(ka)^2(\pi/8)\Phi(ka, b/a, \phi).$$

Evidently, $\Phi(ka, b/a, 0) > 0$. It turns out that Φ remains positive for a range of ϕ. In Fig. 1, we take $b = 2a$ and plot $\phi_0(ka)$, where $\Phi(ka, 2, \phi) > 0$ for $0 \leqslant \phi < \phi_0(ka)$ and $\Phi(ka, 2, \phi_0) = 0$. The two horizontal asymptotes in the figure can be predicted. Thus, $\phi_0(ka) \sim (a/b)^2$ as $ka \to 0$ whereas, when $b = 2a$, $\phi_0(ka) \sim \frac{5}{48}\pi^2[\Gamma(3/4)]^{-4}$ as $ka \to \infty$ [13]. The figure shows that we cannot use Eq. (6.7) for filling fractions that are too large.

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Appendix A. Evaluation of some integrals

A.1. Evaluation of (5.5)

From Eqs. (4.4) and (4.7), we have

$$\int_{\mathrm{disc}'} \mathrm{e}^{\mathrm{i}\kappa x_1} I_1 \,\mathrm{d}x_1\,\mathrm{d}y_1 = -\sum_{n=-\infty}^{\infty} \mathrm{i}^n C_n^{(1)} \int_{\mathrm{disc}'} H_n(kr_1)\mathrm{e}^{\mathrm{i}n\theta_1}\mathrm{e}^{\mathrm{i}\kappa x_1}\,\mathrm{d}x_1\,\mathrm{d}y_1 \tag{A.1}$$

$$= \frac{2\mathrm{i}}{k}\sum_{n=-\infty}^{\infty} C_n^{(1)} \frac{(\kappa + k)\mathrm{e}^{\mathrm{i}kx} + \pi\mathrm{i}k\mathrm{e}^{\mathrm{i}\kappa x}\mathscr{N}_n(\kappa a)}{k^2 - \kappa^2}, \tag{A.2}$$

where $\mathrm{Im}\,\kappa > 0$ and

$$\mathscr{N}_n(\kappa a) = kaH_n'(ka)J_n(\kappa a) - \kappa a H_n(ka)J_n'(\kappa a). \tag{A.3}$$

Here, we have evaluated the integral on the right-hand side of (A.1) as on p. 3419 of [6]; the result is exact when $x > a$. Recall that the region disc′ consists of that part of the half-plane $x_1 > 0$ that is outside the circle $r_1^2 \equiv (x_1 - x)^2 + y_1^2 = a^2$. The method of evaluation in [6] uses Green's theorem to reduce the double integral to the sum of an integral along $x_1 = 0$ and an integral around the circle $r_1 = a$. When $0 < x < a$, this circle cuts the y_1-axis, and then Eq. (A.2) should be regarded as an approximation. More precisely, we have

$$\int_{\mathrm{disc}'} H_n(kr_1)\mathrm{e}^{\mathrm{i}n\theta_1}\mathrm{e}^{\mathrm{i}\kappa x_1}\,\mathrm{d}x_1\,\mathrm{d}y_1 = \frac{2}{\mathrm{i}k}(-\mathrm{i})^n\frac{(\kappa + k)\mathrm{e}^{\mathrm{i}kx} + \pi\mathrm{i}k\mathrm{e}^{\mathrm{i}\kappa x}\mathscr{N}_n(\kappa a)}{k^2 - \kappa^2} + E_n(x),$$

where $E_n(x) = 0$ for $x \geqslant a$,

$$E_n(x) = \int_{\mathscr{D}} H_n(kr_1)\mathrm{e}^{\mathrm{i}n\theta_1}\mathrm{e}^{\mathrm{i}\kappa x_1}\,\mathrm{d}x_1\,\mathrm{d}y_1 \quad \text{for} \quad 0 < x < a$$

and $\mathscr{D}$ is the segment of the disc $r_1 < a$ with $x_1 < 0$.

Now, we let $\kappa \to k$. Put $\kappa = k + i\varepsilon$ with $\varepsilon > 0$. We have $e^{i\kappa x} \simeq (1-\varepsilon x)e^{ikx}$, $k^2 - \kappa^2 \simeq -2i\varepsilon k$ and $\mathscr{N}_n(\kappa a) \simeq 2i/\pi + i\varepsilon k a^2 d_n(ka)$, with d_n defined by Eq. (2.13). Hence, in the limit $\varepsilon \to 0+$, the right-hand side of Eq. (A.2) becomes

$$\frac{e^{ikx}}{ik^2}\sum_{n=-\infty}^{\infty} C_n^{(1)}\{1 - 2ikx + \pi i(ka)^2 d_n(ka)\}. \tag{A.4}$$

Finally, use of Eqs. (2.21), (2.25) and (4.8) gives the result (5.5).

In a similar way, using Eqs. (4.14), (4.16) and (2.23), we obtain

$$\begin{aligned}\lim_{\kappa\to k}\int_{\mathrm{disc}'} e^{i\kappa x_1} I_2\,dx_1\,dy_1 &= \frac{e^{ikx}}{ik^2}\sum_{n=-\infty}^{\infty} C_n^{(2)}\{1 - 2ikx + \pi i(ka)^2 d_n(ka)\}\\ &= -\frac{\pi^2 a^2}{4}(ka)^2 e^{ikx}(1-2ikx)\mathscr{H}(ka) + \pi a^2 e^{ikx}\sum_{n=-\infty}^{\infty} d_n(ka)C_n^{(2)}.\end{aligned} \tag{A.5}$$

A.2. Evaluation of (5.4)

Next, we consider the integral over the disc,

$$\int_{r_1<a} e^{ikx_1} I_1(r_1,\theta_1)dx_1\,dy_1 = \frac{\pi^2}{4i}e^{ikx}\sum_{n=-\infty}^{\infty}\int_0^a \Lambda_n^{(1)} J_n(kr)r\,dr, \tag{A.6}$$

where we have used $x_1 = x - r_1\cos\theta_1$, Eqs. (2.1), (4.4) and (4.11), and integrated over θ_1. Substituting Eq. (4.13), and then using Eqs. (4.10) and (4.19), we obtain

$$\int_0^a \Lambda_n^{(1)}(kr)J_n(kr)r\,dr = a^2(ka)^2 d_n(ka)\mathscr{J}_n(ka) - \frac{2ia^2}{\pi}J_{n-1}(ka)J_{n+1}(ka). \tag{A.7}$$

Finally, making use of Eq. (2.23), we obtain Eq. (5.4).

Similarly, using Eq. (4.17),

$$\int_{r_1<a} e^{ikx_1} I_2(r_1,\theta_1)dx_1\,dy_1 = -\frac{\pi^3}{32}e^{ikx}\sum_{n=-\infty}^{\infty}\int_0^a \Lambda_n^{(2)} J_n(kr)r\,dr = -\pi a^2 e^{ikx}\sum_{n=-\infty}^{\infty} d_n(ka)C_n^{(2)} + \frac{\pi^2 i}{4k^2}e^{ikx}\sum_{n=-\infty}^{\infty}\mathscr{F}_n(ka), \tag{A.8}$$

where we have used Eqs. (4.15) and (4.18),

$$\mathscr{F}_n(ka) = k^2\int_0^a X_n(r)J_n(kr)r\,dr$$

and $X_n(r)$ is given by Eq. (4.20).

Adding Eqs. (A.5) and (A.8), the sums containing $C_n^{(2)}$ cancel leaving

$$\lim_{\kappa\to k}\int_{x_1>0} e^{i\kappa x_1} I_2\,dx_1\,dy_1 = \frac{\pi}{4}a^2 e^{ikx}\{P_0(ka) + (2ikx - 1)Q_0(ka)\}$$

for $x > a$, where

$$P_0(ka) = \frac{\pi i}{(ka)^2}\sum_{n=-\infty}^{\infty}\mathscr{F}_n(ka) \quad \text{and} \quad Q_0(ka) = \pi(ka)^2\mathscr{H}(ka).$$

The expression for P_0 simplifies a little. From Eq. (4.20), $X_n = X_n^{(1)} + X_n^{(2)}$, with $X_n^{(1)}$ and $X_n^{(2)}$ given by Eqs. (4.21) and (4.22), respectively. We have

$$\begin{aligned} k^2\int_0^a r\sum_{n=-\infty}^{\infty} X_n^{(1)} J_n(kr)dr &= k^4\int_0^a r\frac{ir^2}{\pi}dr = \frac{i}{4\pi}(ka)^4,\\ k^2\int_0^a X_n^{(2)} J_n(kr)r\,dr &= \frac{1}{2}(ka)^4\mathscr{J}_n(ka)[J_n(ka)H_n(ka) - d_n(ka) - i/\pi],\\ k^2\sum_{n=-\infty}^{\infty}\int_0^a X_n^{(2)} J_n(kr)r\,dr &= \frac{1}{2}(ka)^4\left[\sum_{n=-\infty}^{\infty}\mathscr{J}_n J_n H_n + 4i\mathscr{H} - \frac{i}{\pi}\right].\end{aligned}$$

Hence

$$P_0(ka) = (ka)^2\left[\frac{1}{4} - 2\pi\mathscr{H}(ka) + \frac{\pi i}{2}\sum_{n=-\infty}^{\infty}\mathscr{J}_n(ka)J_n(ka)H_n(ka)\right]. \tag{A.9}$$

Notice that

$$\operatorname{Im} P_0 = \frac{\pi}{2}(ka)^2 \sum_{n=-\infty}^{\infty} \mathscr{J}_n(ka) J_{n-1}(ka) J_{n+1}(ka) \sim \frac{\pi}{8}(ka)^4 \tag{A.10}$$

as $ka \to 0$, implying that P_0 does not vanish identically. In fact, a more detailed calculation shows that

$$P_0(ka) \sim \frac{1}{4}(ka)^4 \log ka \quad \text{as} \quad ka \to 0. \tag{A.11}$$

Appendix B. An integral representation for $\mathcal{H}(ka)$

We give an integral representation for $\mathscr{H}(ka)$, defined by Eq. (2.25); it is

$$\mathscr{H}(ka) = \frac{1}{|D_0|^2} \int_{D_0} \int_{D_0} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} G_0(\mathbf{r}, \mathbf{r}') \, dV \, dV', \tag{B.1}$$

where $| D_0 |= \pi a^2$ is the area of the disc D_0 and $\mathbf{k}$ is a constant vector with $| \mathbf{k} |= k$. To see this, we put $\mathbf{k} = (k \cos \alpha, k \sin \alpha)$ and then we find that

$$\int_{D_0} e^{i\mathbf{k}\cdot\mathbf{r}} G_0(\mathbf{r}, \mathbf{r}') \, dV = \frac{\pi i}{8k^2} \sum_{n=-\infty}^{\infty} i^n \Lambda_n^{(1)}(kr') e^{in(\alpha-\theta')},$$

where we have used Eqs. (2.1), (4.6) and (4.12). (This calculation is similar to that in Section 4.1.2.)

Then, using Eq. (4.15), the right-hand side of Eq. (B.1) becomes

$$\frac{4i}{\pi^2 (ka)^4} \sum_{n=-\infty}^{\infty} C_n^{(2)};$$

this reduces to $\mathscr{H}(ka)$, once Eqs. (4.16) and (2.23) are used.

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