Scattering matrix properties with evanescent modes for waveguides in fluids and solids

V. Pagneux[a)]
Laboratoire d'Acoustique de l'Université du Maine, UMR CNRS 6613, Av. Olivier Messiaen, 72085 Le Mans Cedex 9, France

A. Maurel[b)]
Laboratoire Ondes et Acoustique, UMR CNRS 7587, Ecole Supérieure de Physique et de Chimie Industrielles, 10 rue Vauquelin, 75005 Paris, France

(Received 23 May 2003; revised 11 June 2004; accepted 24 June 2004)

Reciprocity, energy conservation, and time-reversal invariance are three general properties of the wave fields that imply algebraic scattering matrix properties. In this paper, these scattering matrix properties are established for waveguides when evanescent modes are taken into account. The situations correspond to guided acoustic pressure waves in fluids and Lamb waves in solids treated with the same formalism. The relations between the three properties verified by the scattering matrix are then discussed, and it is found that, as soon as two properties are verified, the third is also verified. [DOI: 10.1121/1.1786293]

PACS numbers: 43.20.Fn, 43.20.Gp, 43.20.Mv [MO] Pages: 1913–1920

I. INTRODUCTION

A very convenient quantity to characterize the scattering of a wave by a scattering region is the so-called scattering matrix (S matrix). The versatility of this formulation is due to the fact that it relates the outgoing wave to the ingoing wave directly, with respect to the scattering region, which are intuitive quantities.[1–4] The scattering problem is then to determine the S matrix and not to determine the wave field in the whole space.[5,6] From an experimental point of view, the S matrix is a very convenient tool since the measurements have to be performed outside of the scattering region only and do not have to be intrusive.

In waveguides, when the interest is only in the far field of the scattering region, the S matrix is restricted to the propagative components. Then, the fundamental properties of the wave propagation (i.e., reciprocity, energy conservation, and time-reversal symmetry) are very simply translated into relationships verified by the S matrix.[7] For instance, the energy conservation implies that the S matrix is unitary with proper normalization. Besides their fundamental interest, these algebraic properties of the S matrix can be very useful in experimental and numerical works, where they provide a convenient way to check if the fundamental properties of the wave propagation are verified. Nevertheless, as soon as the near field is considered, the S matrix has to include the evanescent waves; such a need can exist for example if several scattering regions are taken into account and if they are close enough. Then, the usual properties of the S matrix, correct for the propagative waves only, are not correct anymore.[8,9]

In this paper, we investigate the general relationships verified by the S matrix with evanescent waves in the cases of the propagation in 2D waveguides in fluids or solids. The first case corresponds to the guided propagation of a scalar wave in a fluid with hard wall and the second case corresponds to the guided propagation of either a scalar wave (SH wave) or a vectorial wave (Lamb wave) in an elastic waveguide with free boundaries. We treat the waveguides in fluids and solids with the same formalism that appears to be useful to consider the S matrix.

The plan of the paper is as follows. In Sec. II, we define the scattering matrix for the waveguides. Then, the method used to project the acoustic fields on the waveguide modes is presented in Sec. III. In Sec. IV, we find the three properties verified by the S matrix due to reciprocity, time-reversal symmetry, and energy conservation of the wave propagation, when evanescent modes are taken into account. Finally, in Sec. V, we summarize the three properties and discuss the relation between the three properties of the S matrix: we find that as soon as two properties are verified the third relation is automatically verified.

II. DEFINITION OF THE SCATTERING MATRIX

We consider the problem corresponding to Fig. 1. It consists of a bidimensional waveguide of longitudinal axis x with constant height for $x \leqslant 0$ and $x \geqslant L$. In the scattering region, $0 < x < L$, the waveguide is of variable height $h(x)$. The harmonic time dependence with pulsation ω is $e^{-i\omega\tau}$ and it will be omitted in the following.

Outside of the scattering region, the wave fields can be expressed as a sum over the transverse modes of the homogeneous waveguide with coefficients depending on x. These coefficients can be split into right-going components $\mathbf{A}$ and left-going components $\mathbf{B}$, that is, $\mathbf{A}$ and $\mathbf{B}$ are vectors whose components are the projections of the wave field on the wave modes. For the sake of clarity, we note in the following $\mathbf{A}^I$ (respectively, $\mathbf{B}^I$) at $x=0$ and $\mathbf{A}^{II}$ (respectively, $\mathbf{B}^{II}$) at $x=L$. We define $\mathbf{\Psi}_{\text{in}}$ as the ingoing waves and $\mathbf{\Psi}_{\text{out}}$ as the outgoing waves, with respect to the scattering region

a)Electronic mail: vincent.pagneux@univ-lemans.fr
b)Electronic mail: agnes.maurel@espci.fr

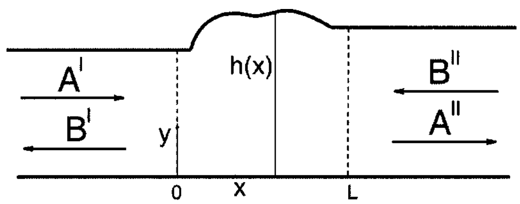

FIG. 1. Geometry of the waveguide.

$$\Psi_{\rm in}=\begin{pmatrix}\mathbf{A}^I\\ \mathbf{B}^{II}\end{pmatrix},\quad \Psi_{\rm out}=\begin{pmatrix}\mathbf{B}^I\\ \mathbf{A}^{II}\end{pmatrix}. \tag{2.1}$$

Then, the scattering matrix links together the outgoing waves $\Psi_{\rm out}$ and the ingoing waves $\Psi_{\rm in}$ at both extremities of the scattering region

$$\Psi_{\rm out}=\mathsf{S}\Psi_{\rm in}. \tag{2.2}$$

Actually, S is defined with the reflection and transmission matrices R^I, T^I (respectively R^{II}, T^{II}) corresponding to wave incident from the left (respectively, from the right)

$$\mathsf{S}=\begin{pmatrix}\mathsf{R}^I & \mathsf{T}^{II}\\ \mathsf{T}^I & \mathsf{R}^{II}\end{pmatrix}. \tag{2.3}$$

Here, R and T are matrices linking ingoing and outgoing wave components. Note that the scattering matrix could be used for the case of more than two terminating waveguides.

III. FORMALISM

A. System

Guided wave propagation in fluids and solids is considered. The fluid case corresponds to the Helmholtz equation on the pressure p associated with boundary condition $\partial_{\mathbf{n}}p=0$ on the walls $y=0$ and $y=h(x)$ ($\mathbf{n}$ denotes the vector normal to the wall). The solid case corresponds to the Navier equation on the displacement vector $\mathbf{w}$ with the boundary condition for a wall free of traction $\sigma\cdot\mathbf{n}=0$ on $y=0$ and $y=h(x)$, where $\sigma=\lambda\nabla\cdot\mathbf{w}I+\mu(\nabla\mathbf{w}+{}^t\nabla\mathbf{w})$ denotes the stress tensor, and (λ,μ) are the Lamé's constants. Results can be easily generalized to inhomogeneous media with variable ρ, λ, and μ and to other boundary conditions, e.g., $p=0$ for fluids or $\mathbf{w}=0$ for solids on the walls.

We choose to present both cases (fluid and solid) in the same formalism. This is done working with two quantities $\mathbf{X}$ and $\mathbf{Y}$ presented below. In addition to permitting a unified presentation, that formalism allows us to easily tackle the projection on the Lamb modes (for propagation in solids). The idea is to write the equations as an evolution equation (with respect to the axis x of the waveguide) on $\mathbf{X}$ and $\mathbf{Y}$ that leads to a canonical eigenvalue problem in the transverse direction when transverse modes are sought as in Sec. III B. For solids, this formulation is similar to the one presented recently in Ref. 10 in that it describes the evolution of a stress-displacement 4-vector, but here that 4-vector is suitably split in two 2-vectors that permit one to project easily on the transverse modes.[11]

For fluids, $\mathbf{X}$ and $\mathbf{Y}$ are scalar quantities defined by

$$\mathbf{X}=\frac{\partial}{\partial x}p \quad \text{and} \quad \mathbf{Y}=p.$$

For solids, they are 2-vectors

$$\mathbf{X}=\begin{pmatrix}u\\ t\end{pmatrix} \quad \text{and} \quad \mathbf{Y}=\begin{pmatrix}-s\\ v\end{pmatrix},$$

where

$$\mathbf{w}=\begin{pmatrix}u\\ v\end{pmatrix}$$

is the displacement vector and

$$\sigma=\begin{pmatrix}s & t\\ t & r\end{pmatrix}$$

the corresponding stress tensor.

It is shown in Appendix A that both sets of equations can be written in the same generic form

$$\frac{\partial}{\partial x}\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix}=\begin{pmatrix}0 & \mathsf{F}\\ \mathsf{G} & 0\end{pmatrix}\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix}, \tag{3.1}$$

with boundary conditions

$$\begin{aligned}&(\mathsf{C}_\mathsf{X}^0+h'\mathsf{C}_\mathsf{X}^1)\mathbf{X}+(\mathsf{C}_\mathsf{Y}^0+h'\mathsf{C}_\mathsf{Y}^1)\mathbf{Y}=0, \quad \text{at } y=h(x),\\ &\mathsf{C}_\mathsf{X}^0\mathbf{X}+\mathsf{C}_\mathsf{Y}^0\mathbf{Y}=0, \quad \text{at } y=0,\end{aligned} \tag{3.2}$$

where $h'=dh/dx$.

Expressions of F, G, C_X^i, and C_Y^i $(i=0,1)$ in both cases are given in Appendix A. In the following, we use two properties of these matrices

(1) (Property 1): F, G, C_X^i, and C_Y^i $(i=0,1)$ are real;
(2) (Property 2): $(\mathsf{F}\mathbf{Y}|\tilde{\mathbf{Y}})+(\mathbf{X}|\mathsf{G}\tilde{\mathbf{X}})=(\mathbf{Y}|\mathsf{F}\tilde{\mathbf{Y}})+(\mathsf{G}\mathbf{X}|\tilde{\mathbf{X}})+h'(\mathbf{Y}\cdot\tilde{\mathbf{X}}-\mathbf{X}\cdot\tilde{\mathbf{Y}})(h)$,

where $(\mathbf{X},\mathbf{Y})$ and $(\tilde{\mathbf{X}},\tilde{\mathbf{Y}})$ are two solutions of Eq. (3.1) with boundary condition (3.2) and $(\mathbf{U}|\mathbf{V})=\int_0^{h(x)}\mathbf{U}(x,y)\cdot\mathbf{V}(x,y)dy$ denotes a bilinear form. Properties 1 and 2 are demonstrated in Appendix B. In Sec. IV, it will appear that property 1 is related to the time-reversal invariance, property 2 is related to the reciprocity, and both properties are related to energy conservation.

It can be noticed that both properties 1 and 2 involve the boundary conditions. For instance in the fluid case, if the walls were lined, property 2 would be conserved while property 1 would not.

B. Modal decomposition

The scattering matrix S links together the right-going components and the left-going components through (2.2). In this section, we expose the modal decomposition that permits us to define the right-going components $\mathbf{A}$ and left-going components $\mathbf{B}$.

The transverse modes used in the decomposition, denoted $\mathbf{X}_n(y)$ and $\mathbf{Y}_n(y)$, correspond to the natural basis in a uniform waveguide. They are associated with a wave number k_n and are defined by

$$ik_n\begin{pmatrix}\mathbf{X}_n\\ \mathbf{Y}_n\end{pmatrix}=\begin{pmatrix}0 & \mathsf{F}\\ \mathsf{G} & 0\end{pmatrix}\begin{pmatrix}\mathbf{X}_n\\ \mathbf{Y}_n\end{pmatrix}, \tag{3.3}$$

with boundary conditions corresponding to Eq. (3.2) setting $h'=0$

$$\mathsf{C}_\mathsf{X}^0\mathbf{X}_n+\mathsf{C}_\mathsf{Y}^0\mathbf{Y}_n=0, \quad \text{at } y=0 \quad \text{and } y=h(x). \tag{3.4}$$

These modes are well known: they are proportional to the cosine functions[12] in the fluid case and are the Lamb modes in the solid case[13] (their expressions are given in Appendix C). They are linked by a biorthogonality relation

$$(\mathbf{X}_n|\mathbf{Y}_m)=\mathcal{J}_n\delta_{nm}. \tag{3.5}$$

This relation corresponds to the orthogonality of the cosine functions in the fluid case and to the Fraser's biorthogonality relation in the solid case.[14] In the fluid case, $\mathcal{J}_n=ik_n$, and in the solid case, its expression can be found in Ref. 11. Here, this biorthogonality relation can be easily demonstrated using (3.3) and the symmetry property of F and G when they are applied to $\mathbf{X}_n$ and $\mathbf{Y}_n$: $(\mathsf{F}\mathbf{Y}_n|\mathbf{Y}_m)=(\mathbf{Y}_n|\mathsf{F}\mathbf{Y}_m)$ and $(\mathsf{G}\mathbf{X}_n|\mathbf{X}_m)=(\mathbf{X}_n|\mathsf{G}\mathbf{X}_m)$. The biorthogonality equation (3.5) is of interest because it permits one to easily project $\mathbf{X}$ and $\mathbf{Y}$ on the modes $\mathbf{X}_n$ and $\mathbf{Y}_n$.

In the following, k_n is indexed by $n>0$ when it refers to a right-going wave and $n<0$ when it refers to a left-going wave. Using the symmetry properties of the basis $\mathbf{X}_{-n}=-\mathbf{X}_n$ and $\mathbf{Y}_{-n}=\mathbf{Y}_n$, $\mathbf{X}$ and $\mathbf{Y}$ are decomposed as

$$\begin{aligned}\mathbf{X}(x,y)&=\sum_{n>0}(A_n(x)-B_n(x))\mathbf{X}_n(y),\\ \mathbf{Y}(x,y)&=\sum_{n>0}(A_n(x)+B_n(x))\mathbf{Y}_n(y).\end{aligned} \tag{3.6}$$

The S matrix is concerned with values of the components $\mathbf{A}$ and $\mathbf{B}$ at $x=0$ and $x=L$ (Fig. 1). At these two x positions, the values of $\mathbf{X}_n$, $\mathbf{Y}_n$, and $\mathcal{J}_n$ [Eq. (3.5)] are *a priori* distinct and we note in the following $\mathbf{X}_n^\alpha$, $\mathbf{Y}_n^\alpha$, and $\mathcal{J}_n^\alpha$ with $\alpha=I$, II for $\mathbf{X}_n$, $\mathbf{Y}_n$, and $\mathcal{J}_n$ on the cross section $x=0$, L, respectively. At each cross section, we define the matrices

$$\begin{aligned}\mathsf{J}_{mn}^I&=(\mathbf{X}_n^I|\mathbf{Y}_m^I)=\mathcal{J}_n^I\delta_{mn},\\ \tilde{\mathsf{J}}_{mn}^I&=(\mathbf{X}_n^I|\overline{\mathbf{Y}_m^I}),\end{aligned} \tag{3.7}$$

and the same for J^{II} and $\tilde{\mathsf{J}}^{II}$. Then, J and M are such that

$$\mathsf{J}=\begin{pmatrix}\mathsf{J}^I & 0\\ 0 & \mathsf{J}^{II}\end{pmatrix}, \quad \mathsf{JM}=\begin{pmatrix}\tilde{\mathsf{J}}^I & 0\\ 0 & \tilde{\mathsf{J}}^{II}\end{pmatrix}. \tag{3.8}$$

By definition, J is a diagonal matrix [Eq. (3.7)]. In the fluid case, M is simply the identity matrix while in the solid case, M has a more complicated structure, as illustrated in Sec. V and detailed in Appendix C.

IV. PROPERTIES OF THE SCATTERING MATRIX

A. Reciprocity

The reciprocity relation corresponds to a relation between two solutions $(\mathbf{X},\mathbf{Y})$ and $(\tilde{\mathbf{X}},\tilde{\mathbf{Y}})$ of (3.1)–(3.2). We show below that this relation can be written

$$(\mathbf{X}^I|\tilde{\mathbf{Y}}^I)-(\tilde{\mathbf{X}}^I|\mathbf{Y}^I)=(\mathbf{X}^{II}|\tilde{\mathbf{Y}}^{II})-(\tilde{\mathbf{X}}^{II}|\mathbf{Y}^{II}). \tag{4.1}$$

This comes from

$$\begin{aligned}&d_x((\mathbf{X}|\tilde{\mathbf{Y}})-(\tilde{\mathbf{X}}|\mathbf{Y}))\\ &=\left(\frac{\partial}{\partial x}\mathbf{X}\middle|\tilde{\mathbf{Y}}\right)+\left(\mathbf{X}\middle|\frac{\partial}{\partial x}\tilde{\mathbf{Y}}\right)-\left(\frac{\partial}{\partial x}\tilde{\mathbf{X}}\middle|\mathbf{Y}\right)-\left(\tilde{\mathbf{X}}\middle|\frac{\partial}{\partial x}\mathbf{Y}\right)\\ &\quad+h'(\mathbf{X}\tilde{\mathbf{Y}}-\tilde{\mathbf{X}}\mathbf{Y})(h)=(\mathsf{F}\mathbf{Y}|\tilde{\mathbf{Y}})+(\mathbf{X}|\mathsf{G}\tilde{\mathbf{X}})-(\mathsf{F}\tilde{\mathbf{Y}}|\mathbf{Y})\\ &\quad-(\tilde{\mathbf{X}}|\mathsf{G}\mathbf{X})+h'(\mathbf{X}\tilde{\mathbf{Y}}-\tilde{\mathbf{X}}\mathbf{Y})(h)=0.\end{aligned}$$

The result is deduced from property 2 since $(\mathbf{X},\mathbf{Y})$ and $(\tilde{\mathbf{X}},\tilde{\mathbf{Y}})$ are solutions of (3.1)–(3.2). This form of the reciprocity relation is similar to the one found in Ref. 15 in the fluid case, where it is called a reciprocity theorem of the convolution type. Obviously, Eq. (4.1) can also be deduced from the usual integral representation of the reciprocity property.

Using the modal decomposition (3.6) and the biorthogonality (3.5), the reciprocity relation (4.1) takes the form

$${}^t\mathbf{A}^I\mathsf{J}^I\tilde{\mathbf{B}}^I+{}^t\mathbf{B}^{II}\mathsf{J}^{II}\tilde{\mathbf{A}}^{II}-{}^t\mathbf{B}^I\mathsf{J}^I\tilde{\mathbf{A}}^I-{}^t\mathbf{A}^{II}\mathsf{J}^{II}\tilde{\mathbf{B}}^{II}=0.$$

Here, we have used $\mathsf{J}^\alpha={}^t\mathsf{J}^\alpha$, $\alpha=I$, II. With Eqs. (2.1) and (3.8), this latter expression is equivalent to

$${}^t\mathbf{\Psi}_{\text{in}}\mathsf{J}\tilde{\mathbf{\Psi}}_{\text{out}}-{}^t\mathbf{\Psi}_{\text{out}}\mathsf{J}\tilde{\Psi}_{\text{in}}=0.$$

Eventually, using the scattering matrix S, we obtain

$$\mathsf{JS}-{}^t\mathsf{SJ}=0, \tag{4.2}$$

which is the property of the scattering matrix induced by reciprocity of the propagation in the scattering region. It has to be noted that Eq. (4.2) has the same form with or without evanescent modes.

B. Time-reversal invariance

In both cases, fluid and solid, matrices F, G, C_X^i, and C_Y^i $(i=0,1)$ in (3.1) are real (property 1). As a consequence, if $(\mathbf{X},\mathbf{Y})$ is a solution of (3.1), then $(\overline{\mathbf{X}},\overline{\mathbf{Y}})$ is also a solution of (3.1). This solution corresponds to the time-reversed solution, noted $(\mathbf{X}^R,\mathbf{Y}^R)$ in the following. Thus, the time-reversal invariance is translated in the harmonic regime by

$$\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix}\text{solution}\Rightarrow\begin{pmatrix}\mathbf{X}^R=\overline{\mathbf{X}}\\ \mathbf{Y}^R=\overline{\mathbf{Y}}\end{pmatrix}\text{solution}. \tag{4.3}$$

To obtain a property for the S matrix from the time reversal, the idea is to use $\mathbf{\Psi}_{\text{out}}^R=\mathsf{S}\mathbf{\Psi}_{\text{in}}^R$ and to express $\mathbf{\Psi}_{\text{out}}^R$ and $\mathbf{\Psi}_{\text{in}}^R$ with $\overline{\mathbf{\Psi}}_{\text{in}}$ as $\mathbf{\Psi}_{\text{in}}^R=K(\mathsf{S})\overline{\mathbf{\Psi}}_{\text{in}}$ and $\mathbf{\Psi}_{\text{out}}^R=L(\mathsf{S})\overline{\mathbf{\Psi}}_{\text{in}}$; thereafter, the relation $L(\mathsf{S})=\mathsf{S}K(\mathsf{S})$ is deduced. We detail this calculation below.

According to the modal decomposition, we have (3.6)

$$\begin{aligned}&\mathbf{X}^R=\sum_n(A_n^R-B_n^R)X_n, \quad\text{and}\quad \mathbf{Y}^R=\sum_n(A_n^R+B_n^R)Y_n,\\ &\overline{\mathbf{X}}=\sum_n(\overline{A_n}-\overline{B_n})\overline{X_n}, \quad\text{and}\quad \overline{\mathbf{Y}}=\sum_n(\overline{A_n}+\overline{B_n})\overline{Y_n}.\end{aligned} \tag{4.4}$$

With $\mathbf{X}^R=\overline{\mathbf{X}}$ and $\mathbf{Y}^R=\overline{\mathbf{Y}}$, the equalities $(\mathbf{X}^R|\mathbf{Y}_n)=(\overline{\mathbf{X}}|\mathbf{Y}_n)$ and $(\mathbf{Y}^R|\mathbf{X}_n)=(\overline{\mathbf{Y}}|\mathbf{X}_n)$ become, using (3.5) and (4.4)

$$(A_n^R - B_n^R)\mathcal{J}_n = \sum_{m>0} (\overline{A_m} - \overline{B_m})(\overline{\mathbf{X}_m}|\mathbf{Y}_n),$$

$$(A_n^R + B_n^R)\mathcal{J}_n = \sum_{m>0} (\overline{A_m} + \overline{B_m})(\overline{\mathbf{Y}_m}|\mathbf{X}_n).$$

Summing and subtracting these relations, at $x=0$ $(\alpha=I)$ and at $x=L$ $(\alpha=II)$, leads to

$$2\mathsf{J}^\alpha \mathbf{A}^{R,\alpha} = (\tilde{\mathsf{J}}^\alpha + {}^t\overline{\tilde{\mathsf{J}}^\alpha})\overline{\mathbf{A}^\alpha} + (\tilde{\mathsf{J}}^\alpha - {}^t\overline{\tilde{\mathsf{J}}^\alpha})\overline{\mathbf{B}^\alpha},$$

$$2\mathsf{J}^\alpha \mathbf{B}^{R,\alpha} = (\tilde{\mathsf{J}}^\alpha - {}^t\overline{\tilde{\mathsf{J}}^\alpha})\overline{\mathbf{A}^\alpha} + (\tilde{\mathsf{J}}^\alpha + {}^t\overline{\tilde{\mathsf{J}}^\alpha})\overline{\mathbf{B}^\alpha},$$

for $\alpha=I$, II. Using the definitions (2.1) and (2.2), these relations become

$$2\mathsf{J}\boldsymbol{\Psi}_{\text{in}}^R = \mathsf{H}_p\overline{\boldsymbol{\Psi}_{\text{in}}} + \mathsf{H}_m\overline{\boldsymbol{\Psi}_{\text{out}}} = (\mathsf{H}_p + \mathsf{H}_m\overline{\mathsf{S}})\overline{\boldsymbol{\Psi}_{\text{in}}},$$

$$2\mathsf{J}\boldsymbol{\Psi}_{\text{out}}^R = \mathsf{H}_m\overline{\boldsymbol{\Psi}_{\text{in}}} + \mathsf{H}_p\overline{\boldsymbol{\Psi}_{\text{out}}} = (\mathsf{H}_m + \mathsf{H}_p\overline{\mathsf{S}})\overline{\boldsymbol{\Psi}_{\text{in}}},$$

where $\mathsf{H}_p = \mathsf{JM} + {}^t(\overline{\mathsf{JM}})$, $\mathsf{H}_m = \mathsf{JM} - {}^t(\overline{\mathsf{JM}})$, and $\overline{\boldsymbol{\Psi}_{\text{out}}} = \overline{\mathsf{S}}\overline{\boldsymbol{\Psi}_{\text{in}}}$ have been used.

Finally, knowing that $\boldsymbol{\Psi}_{\text{out}}^R = \mathsf{S}\boldsymbol{\Psi}_{\text{in}}^R$, we obtain the time-reversal invariance property for the scattering matrix

$$\mathsf{JSJ}^{-1}(\mathsf{H}_p + \mathsf{H}_m\overline{\mathsf{S}}) = \mathsf{H}_m + \mathsf{H}_p\overline{\mathsf{S}}. \tag{4.5}$$

This time-reversal invariance property of the S matrix with evanescent modes is different from the version without evanescent modes because of the extra terms involving H_p (see Sec. V).

C. Energy conservation

The energy conservation comes directly from the reciprocity relation and time-reversal invariance [taking in the reciprocity relation $(\mathbf{X},\mathbf{Y})$ and $(\mathbf{X}^R = \overline{\mathbf{X}}, \mathbf{Y}^R = \overline{\mathbf{Y}})$ as solutions]

$$(\mathbf{X}^I|\overline{\mathbf{Y}^I}) - (\overline{\mathbf{X}^I}|\mathbf{Y}^I) = (\mathbf{X}^{II}|\overline{\mathbf{Y}^{II}}) - (\overline{\mathbf{X}^{II}}|\mathbf{Y}^{II}). \tag{4.6}$$

This relation can be expressed, for $\alpha=I$, II, as the conservation of the energy flux proportional to $W_\alpha = (\mathbf{X}^\alpha|\overline{\mathbf{Y}^\alpha}) - (\overline{\mathbf{X}^\alpha}|\mathbf{Y}^\alpha)$

$$W_\alpha = {}^t\mathbf{A}^\alpha(\tilde{\mathsf{J}}^\alpha - {}^t\overline{\tilde{\mathsf{J}}^\alpha})\overline{\mathbf{A}^\alpha} - {}^t\mathbf{B}^\alpha(\tilde{\mathsf{J}}^\alpha + {}^t\overline{\tilde{\mathsf{J}}^\alpha})\overline{\mathbf{A}^\alpha}$$
$$+ {}^t\mathbf{A}^\alpha(\tilde{\mathsf{J}}^\alpha + {}^t\overline{\tilde{\mathsf{J}}^\alpha})\overline{\mathbf{B}^\alpha} - {}^t\mathbf{B}^\alpha(\tilde{\mathsf{J}}^\alpha - {}^t\overline{\tilde{\mathsf{J}}^\alpha})\overline{\mathbf{B}^\alpha}$$
$$= \text{constant}.$$

This can be written

$${}^t\boldsymbol{\Psi}_{\text{in}}\mathsf{H}_m\overline{\boldsymbol{\Psi}_{\text{in}}} + {}^t\boldsymbol{\Psi}_{\text{in}}\mathsf{H}_p\overline{\boldsymbol{\Psi}_{\text{out}}} - {}^t\boldsymbol{\Psi}_{\text{out}}\mathsf{H}_p\overline{\boldsymbol{\Psi}_{\text{in}}} - {}^t\boldsymbol{\Psi}_{\text{out}}\mathsf{H}_m\overline{\boldsymbol{\Psi}_{\text{out}}} = 0.$$

Finally, we obtain the energy conservation property for the scattering matrix

$$\mathsf{H}_m + \mathsf{H}_p\overline{\mathsf{S}} = {}^t\mathsf{S}(\mathsf{H}_p + \mathsf{H}_m\overline{\mathsf{S}}), \tag{4.7}$$

where H_p implies extra terms due to the presence of evanescent modes.

As for the reciprocity relation, Eq. (4.6) can also be deduced from an integral representation. It would permit one to generalize Eq. (4.6) to geometry with more than two terminating waveguides.

V. DISCUSSION

We have seen in both cases (fluid and solid) that the reciprocity relation, the time-reversal invariance, and the energy conservation lead to three relations on the scattering matrix S

$$\text{Reciprocity relation:} \quad \mathsf{JS} - {}^t\mathsf{SJ} = 0, \tag{5.1a}$$

Time-reversal invariance:

$$\mathsf{H}_m + \mathsf{H}_p\overline{\mathsf{S}} = \mathsf{JSJ}^{-1}(\mathsf{H}_p + \mathsf{H}_m\overline{\mathsf{S}}), \tag{5.1b}$$

$$\text{Energy conservation:} \quad \mathsf{H}_m + \mathsf{H}_p\overline{\mathsf{S}} = {}^t\mathsf{S}(\mathsf{H}_p + \mathsf{H}_m\overline{\mathsf{S}}). \tag{5.1c}$$

In these relations, the presence of evanescent modes is taken into account by the H_p matrix. Notably, the term involving H_p in the energy conservation represents the energy flux carried by evanescent modes.

Note that the same kind of relations have been shown in Ref. 9 in the angular spectrum representation, but in the scalar case only.

A. Structure of H_m and H_p

To gain some hints of the structures of H_p and H_m, let us take an example in both cases. In the fluid case, let us assume that we have, at $x=0$, one propagating mode $\mathcal{J}_1^I = jk_1^I$ (k_1^I real) and we account for one evanescent mode k_2^I (purely imaginary); at $x=L$, we suppose that we have two propagating modes k_1^{II} and k_2^{II} and we account for one evanescent mode k_3^{II}. With $\mathsf{M}=I$ (Appendix C), we have thus

$$\mathsf{H}_p = 2i\begin{pmatrix} 0 & & & & \\ & k_2^I & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & k_3^{II} \end{pmatrix},$$
$$\tag{5.2}$$
$$\mathsf{H}_m = 2i\begin{pmatrix} k_1^I & & & & \\ & 0 & & & \\ & & k_1^{II} & & \\ & & & k_2^{II} & \\ & & & & 0 \end{pmatrix}.$$

In the solid case, properties of $\mathcal{J}$ are given in Appendix B. Let us assume that, at $x=0$, we have one propagating mode ($\mathcal{J}_1^I$ purely imaginary) and we account for two evanescent modes such that $\mathcal{J}_3^I = \overline{\mathcal{J}_2^I}$; at $x=L$, we assume that we have only one propagating mode and one evanescent mode with a purely real $\mathcal{J}_2^{II}$. In this case, M takes the form

$$
\mathsf{M}=\begin{pmatrix} 1 & & & & \\ & 0 & 1 & & \\ & 1 & 0 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}. \tag{5.3}
$$

Thus, we have

$$
\begin{aligned}
\mathsf{H}_p &= 2\begin{pmatrix} 0 & & & & \\ & 0 & \mathcal{J}_2^I & & \\ & \overline{\mathcal{J}_2^I} & 0 & & \\ & & & 0 & \\ & & & & \mathcal{J}_2^{II} \end{pmatrix}, \\
\mathsf{H}_m &= 2\begin{pmatrix} \mathcal{J}_1^I & & & & \\ & 0 & 0 & & \\ & 0 & 0 & & \\ & & & \mathcal{J}_1^{II} & \\ & & & & 0 \end{pmatrix}.
\end{aligned} \tag{5.4}
$$

B. Relationships without evanescent modes

A consequence of the results of the preceding section is that the three relationships on the S matrix can be simplified to

Reciprocity relation: $\mathsf{JS}-{}^t\mathsf{SJ}=0,$ (5.5a)

Time-reversal invariance: $\mathsf{H}_m=\mathsf{JSJ}^{-1}\mathsf{H}_m\bar{\mathsf{S}},$ (5.5b)

Energy conservation: $\mathsf{H}_m={}^t\mathsf{SH}_m\bar{\mathsf{S}},$ (5.5c)

when the modes are restricted to propagating components, since in that case H_p is zero. Then, as it can be expected by a proper normalization, the reciprocity property implies that S is symmetric and the energy conservation implies that S is unitary.

C. Relations between reciprocity, time-reversal, and energy conservation

So far it has been assumed that the propagation is governed by the equation (3.1) with boundary conditions (3.2). It corresponds to the Helmholtz equation for fluids bounded by rigid walls or to the Navier equation for solids with traction free boundaries. Nevertheless, even if the governing equations in the scattering region are unknown, each equation in (5.1) can be used as a test of the corresponding physical property.

It is clear that if the reciprocity relation is verified ($\mathsf{JSJ}^{-1}={}^t\mathsf{S},$), then time-reversal invariance and energy conservation are equivalent (i.e., time-reversal invariance implies energy conservation and vice versa).

It is more difficult to prove that the reciprocity relation comes from time-reversal invariance and energy conservation. To do that, we use the following procedure. The complex conjugate of the time-reversal invariance is written in the form

$$
\overline{\mathsf{SJ}}^{-1}(\overline{\mathsf{H}_p}+\overline{\mathsf{H}_m}\mathsf{S})=\bar{\mathsf{J}}^{-1}(\overline{\mathsf{H}_m}+\overline{\mathsf{H}_p}\mathsf{S}). \tag{5.6}
$$

Then, the energy conservation is written

$$
(\mathsf{H}_p-{}^t\mathsf{SH}_m)\bar{\mathsf{S}}=({}^t\mathsf{SH}_p-\mathsf{H}_m). \tag{5.7}
$$

Finally, using (5.6) in (5.7) to eliminate $\bar{\mathsf{S}}$ yields a relation between ${}^t\mathsf{S}$ and S

$$
\begin{aligned}
&(\mathsf{H}_p-{}^t\mathsf{SH}_m)\bar{\mathsf{J}}^{-1}(\bar{\mathsf{H}}_m+\bar{\mathsf{H}}_p\mathsf{S}) \\
&\quad=({}^t\mathsf{SH}_p-\mathsf{H}_m)\bar{\mathsf{J}}^{-1}(\bar{\mathsf{H}}_p+\bar{\mathsf{H}}_m\mathsf{S}).
\end{aligned} \tag{5.8}
$$

Thus, we have

$$
{}^t\mathsf{SH}_a\mathsf{S}-\mathsf{H}_a+{}^t\mathsf{SH}_b-\mathsf{H}_b\mathsf{S}=0, \tag{5.9}
$$

with

$$
\mathsf{H}_a=\mathsf{H}_p\bar{\mathsf{J}}^{-1}\overline{\mathsf{H}_m}+\mathsf{H}_m\bar{\mathsf{J}}^{-1}\overline{\mathsf{H}_p}, \tag{5.10}
$$

$$
\mathsf{H}_b=\mathsf{H}_p\bar{\mathsf{J}}^{-1}\overline{\mathsf{H}_p}+\mathsf{H}_m\bar{\mathsf{J}}^{-1}\overline{\mathsf{H}_p}. \tag{5.11}
$$

Using that, by construction, J is diagonal, we obtain a simple expression of H_a and H_b

$$
\mathsf{H}_a=2(\mathsf{JM}\bar{\mathsf{M}}-{}^t(\mathsf{M}\bar{\mathsf{M}})\mathsf{J}), \tag{5.12}
$$

$$
\mathsf{H}_b=2(\mathsf{JM}\bar{\mathsf{M}}+{}^t(\mathsf{M}\bar{\mathsf{M}})\mathsf{J}). \tag{5.13}
$$

It is shown in Appendix C that, in both fluid and solid cases, $\mathsf{MM}=\mathrm{I}$. As a consequence, we find $\mathsf{H}_a=0$ and $\mathsf{H}_b=4\mathsf{J}$, so that Eq. (5.9) corresponds to the reciprocity relation.

We have demonstrated that as soon as two relations of (5.1) are verified, the third relation is also verified.

VI. CONCLUDING REMARKS

The main results of this paper are the relationships in Eq. (5.1) that are verified by the scattering matrix with evanescent modes. To the best of our knowledge, this is the first time that such general equations are found for the fluid case and the solid case. These results can be easily generalized to other boundary conditions, to waveguide with inhomogeneous media (λ and μ variable for instance), and to 3D geometry. They are exact whatever the distance from the scattering region, in contrast to usual relations involving only the propagating modes and assuming that one is in the far field of the scattering region. The equation (5.1) can be useful, in a numerical calculation, to test the energy conservation, time-reversal invariance, or the reciprocity of a scattering region. They could be useful also to test the results of an experiment if this latter is able to provide the evanescent components.

Besides, with evanescent modes taken into account, we have shown that, as soon as two physical properties among energy conservation, time-reversal invariance, and reciprocity are verified, the third is also verified. Consequently, this reduces the number of tests that have to be conducted with (5.1). These relationships can be also helpful to works related to time-reversal mirror in waveguides.[17]

APPENDIX A: EVOLUTION EQUATIONS IN WAVEGUIDES

1. Fluid case

We start from the Helmholtz equation

$$\nabla^2 p + \frac{\omega^2}{c^2} p = 0, \tag{A1}$$

where p denotes the acoustic pressure and c the acoustic wave speed in the considered fluid. In an x-axis waveguide of height $h(x)$, the boundary condition $\partial_{\mathbf{n}} p = 0$ (with $\mathbf{n}$ the normal to the wall) can be written

$$\begin{aligned} &-h'(x)\frac{\partial}{\partial x}p + \frac{\partial}{\partial y}p = 0, \quad \text{at } y = h(x), \\ &\frac{\partial}{\partial y}p = 0, \quad \text{at } y = 0. \end{aligned} \tag{A2}$$

With $\mathbf{X} = (\partial/\partial x)p$ and $\mathbf{Y} = p$, (A1) takes the form

$$\frac{\partial}{\partial x}\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix} = \begin{pmatrix} 0 & -\left(\frac{\partial^2}{\partial y^2} + \omega^2/c^2\right) \\ 1 & 0 \end{pmatrix}\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix}, \tag{A3}$$

and the boundary condition (A2)

$$\begin{aligned} &-h'\mathbf{X} + \frac{\partial}{\partial y}\mathbf{Y} = 0, \quad \text{at } y = h(x) \quad \text{and} \\ &\frac{\partial}{\partial y}\mathbf{Y} = 0, \quad \text{at } y = 0. \end{aligned} \tag{A4}$$

We thus identify $\mathsf{F} = -[(\partial^2/\partial y^2) + \omega^2/c^2]$, $\mathsf{G} = 1$, $\mathsf{C}_\mathsf{X}^0 = 0$, $\mathsf{C}_\mathsf{X}^1 = -1$, $\mathsf{C}_\mathsf{Y}^0 = \partial/\partial y$, and $\mathsf{C}_\mathsf{Y}^1 = 0$.

2. Solid case

With (u,v) the vector of displacements and σ the stress tensor, the elasticity equation can be written

$$-\rho\omega^2\begin{pmatrix}u\\ v\end{pmatrix} = \operatorname{div}\sigma, \tag{A5}$$

where

$$\sigma = \begin{pmatrix} s & t \\ t & r \end{pmatrix}, \quad \text{with} \quad \begin{cases} s = \lambda\frac{\partial}{\partial y}v + (\lambda + 2\mu)\frac{\partial}{\partial x}u, \\ t = \mu\left(\frac{\partial}{\partial y}u + \frac{\partial}{\partial x}v\right), \\ r = (\lambda + 2\mu)\frac{\partial}{\partial y}v + \lambda\frac{\partial}{\partial x}u, \end{cases} \tag{A6}$$

where ρ denotes the density, (λ,μ) the Lamé's constants. Boundary condition of a wall free of traction $\sigma\cdot\mathbf{n} = 0$ ($\mathbf{n}$ denotes the normal to the wall) in an x-axis waveguide of height $h(x)$ can be written

$$\begin{aligned} &-h's + t = 0, \quad -h't + r = 0 \quad \text{at } y = h(x), \\ &t = 0, \quad r = 0 \quad \text{at } y = 0. \end{aligned} \tag{A7}$$

With $\mathbf{X} = \binom{u}{t}$ and $\mathbf{Y} = \binom{-s}{v}$, (A5)–(A6) take the form

$$\frac{\partial}{\partial x}\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{\beta}{\lambda} & -\beta\frac{\partial}{\partial y} \\ 0 & 0 & \beta\frac{\partial}{\partial y} & -\left(\rho\omega^2 + \alpha\frac{\partial^2}{\partial y^2}\right) \\ \rho\omega^2 & \frac{\partial}{\partial y} & 0 & 0 \\ -\frac{\partial}{\partial y} & \frac{1}{\mu} & 0 & 0 \end{pmatrix} \times \begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix}, \tag{A8}$$

with $\alpha = 4\mu(\lambda+\mu)/(\lambda+2\mu)$ and $\beta = \lambda/(\lambda+2\mu)$. With $r = \alpha(\partial/\partial y)v + \beta s$, boundary conditions (A7) can be written

$$\begin{pmatrix} 0 & 1 \\ 0 & -h' \end{pmatrix}\mathbf{X} + \begin{pmatrix} h' & 0 \\ -\beta & \alpha\frac{\partial}{\partial y} \end{pmatrix}\mathbf{Y} = 0 \quad \text{at } y = h(x), \tag{A9}$$

and the same with $h' = 0$ at $y = 0$.

We thus identify

$$\mathsf{F} = \begin{pmatrix} -\frac{\beta}{\lambda} & -\beta\frac{\partial}{\partial y} \\ \beta\frac{\partial}{\partial y} & -\left(\rho\omega^2 + \alpha\frac{\partial^2}{\partial y^2}\right) \end{pmatrix},$$

$$\mathsf{G} = \begin{pmatrix} \rho\omega^2 & \frac{\partial}{\partial y} \\ -\frac{\partial}{\partial y} & \frac{1}{\mu} \end{pmatrix}, \quad \mathsf{C}_\mathsf{X}^0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\mathsf{C}_\mathsf{X}^1 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathsf{C}_\mathsf{Y}^0 = \begin{pmatrix} 0 & 0 \\ -\beta & \alpha\frac{\partial}{\partial y} \end{pmatrix}, \quad \text{and}$$

$$\mathsf{C}_\mathsf{Y}^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

APPENDIX B: PROPERTIES OF F AND G

Property 1 is obviously verified in both fluid and solid cases, the expressions of F and G being given in Appendix A.

To obtain property 2, we calculate the products $(\mathsf{F}\mathbf{Y}|\tilde{\mathbf{Y}})$ and $(\mathsf{G}\mathbf{X}|\tilde{\mathbf{X}})$, where $(\mathbf{X},\mathbf{Y})$ and $(\tilde{\mathbf{X}},\tilde{\mathbf{Y}})$ verify relations (A8)–(A9). We have to distinguish here the fluid and solid cases, as detailed below.

1. Fluid case

It is obvious to see from the expressions of F and G given in Appendix A that

$$(\mathsf{F}\mathbf{Y}|\tilde{\mathbf{Y}}) = (\mathbf{Y}|\mathsf{F}\tilde{\mathbf{Y}}) + h'(\mathbf{Y}\tilde{\mathbf{X}} - \mathbf{X}\tilde{\mathbf{Y}})(h),$$

$$(\mathsf{G}\mathbf{X}|\tilde{\mathbf{X}}) = (\mathbf{X}|\mathsf{G}\tilde{\mathbf{X}}),$$

obtained simply by integration by parts and by using (A4). Subtracting these two relations, Property 2 is then clearly verified.

2. Solid case

We denote here (Z_1, Z_2) the two components of a vector $\mathbf{Z}=\mathbf{X}, \tilde{\mathbf{X}}, \mathbf{Y}, \tilde{\mathbf{Y}}$. We can calculate by integrating by parts

$$(\mathsf{F}\mathbf{Y}|\tilde{\mathbf{Y}}) = (\mathbf{Y}|\mathsf{F}\tilde{\mathbf{Y}}) + \left[Y_2\left(-\beta \tilde{Y}_1 + \alpha \frac{\partial}{\partial y} \tilde{Y}_2 \right) - \tilde{Y}_2\left(-\beta Y_1 + \alpha \frac{\partial}{\partial y} Y_2 \right) \right]_0^h,$$

$$(\mathsf{G}\mathbf{X}|\tilde{\mathbf{X}}) = (\mathbf{X}|\mathsf{G}\tilde{\mathbf{X}}) + [X_2 \tilde{X}_1 - X_1 \tilde{X}_2]_0^h,$$

which are valid for any vectors $\mathbf{X}, \tilde{\mathbf{X}}, \mathbf{Y}, \tilde{\mathbf{Y}}$. Using the boundary conditions (A9), we finally write these relations

$$(\mathsf{F}\mathbf{Y}|\tilde{\mathbf{Y}}) = (\mathbf{Y}|\mathsf{F}\tilde{\mathbf{Y}}) + h'(Y_2 \tilde{X}_2 - X_2 \tilde{Y}_2)(h),$$

$$(\mathsf{G}\mathbf{X}|\tilde{\mathbf{X}}) = (\mathbf{X}|\mathsf{G}\tilde{\mathbf{X}}) - h'(Y_1 \tilde{X}_1 - X_1 \tilde{Y}_1)(h).$$

Consequently, we have

$$(\mathsf{F}\mathbf{Y}|\tilde{\mathbf{Y}}) + (\mathbf{X}|\mathsf{G}\tilde{\mathbf{X}}) = (\mathbf{Y}|\mathsf{F}\tilde{\mathbf{Y}}) + (\mathsf{G}\mathbf{X}|\tilde{\mathbf{X}}) + h'(\mathbf{Y}\cdot\tilde{\mathbf{X}} - \mathbf{X}\cdot\tilde{\mathbf{Y}})(h), \tag{B1}$$

which constitutes property 2.

APPENDIX C: TRANSVERSE MODES AND STRUCTURE OF THE M MATRIX

1. Fluid case

The transverse modes, solutions of (3.3) with boundary conditions (3.4) are (for $|n| \geqslant 1$)

$$\mathbf{Y}_n(y) = \sqrt{\frac{2-\delta_{n1}}{h}} \cos\left(\frac{(n-1)\pi y}{h} \right), \quad \mathbf{X}_n(y) = ik_n \mathbf{Y}_n. \tag{C1}$$

Thus, we calculate for $n \geqslant 1$

$$(\mathbf{X}_n|\mathbf{Y}_m) = (\mathbf{X}_n|\overline{\mathbf{Y}}_m) = ik_n \delta_{nm}, \tag{C2}$$

from which it is easy to deduce that $\mathsf{M}=\mathsf{I}$, where M is defined in (3.8).

2. Solid case

For the sake of clarity, only symmetric Lamb modes are presented (for antisymmetric modes, see for instance Ref. 16). Then, $\mathbf{X}_n = \binom{T_n}{U_n}$ and $\mathbf{Y}_n = \binom{V_n}{-S_n}$, where

$$\begin{aligned} U_n &= ik_n \phi_n + \frac{\partial}{\partial y} \psi_n, \\ V_n &= \frac{\partial}{\partial y} \phi_n - ik_n \psi_n, \\ S_n &= \mu\left[-(k_n^2 + 2\beta_n^2 - \alpha_n^2)\phi_n + 2ik_n \frac{\partial}{\partial y} \psi_n \right], \\ T_n &= \mu\left[2ik_n \frac{\partial}{\partial y} \phi_n + (k_n^2 + \alpha_n^2)\psi_n \right], \end{aligned} \tag{C3}$$

defined with the scalar potential $\phi_n = (k_n^2 + \alpha_n^2)\cosh(\beta_n y)\sinh(\alpha_n h)$ and the potential vector $(0,0,\psi_n)$, with $\psi_n(x,y) = -2ik_n\beta_n \sinh(\alpha_n y)\sinh(\beta_n h)$ and with $\alpha_n = (k_n^2 - k_t^2)^{1/2}$, $\beta_n = (k_n^2 - k_l^2)^{1/2}$, $k_t = \omega/c_t = (\rho/\mu)^{1/2}\omega$, and $k_l = \omega/c_l = (\rho/(\lambda + 2\mu))^{1/2}\omega$.

The structure of the spectrum in the solid case is quite more complicated than in the fluid case. The dispersion relation for symmetric modes is of the form[16]

$$\frac{(\alpha_n^2 + k_n^2)^2}{\alpha_n} \sinh(\alpha_n h)\cosh(\beta_n h) - 4k_n^2 \beta_n \sinh(\beta_n h)\cosh(\alpha_n h) = 0. \tag{C4}$$

We refer to a previous paper where $\mathcal{J}_n$ has been explicitly calculated[11] and we summarize below the main results we can extract from its calculation (results are considered for right-going waves, associated with wave number k_n, $\mathrm{Imag}(k_n) \geqslant 0$ in convention $e^{-i\omega\tau}$).

(i) Propagating modes correspond to real wave number k_n; in this case, $\mathcal{J}_n$ is purely imaginary.

(ii) Evanescent modes that correspond to complex wave number k_n with a nonzero real part give a imaginary $\mathcal{J}_n$ with nonzero imaginary part. Such a mode with k_n wave number can be associated with another one such that $k_m = -\overline{k_n}$. In this case, we have $\mathcal{J}_m = \overline{\mathcal{J}_n}$ and we choose a numbering such that $m = n+1$.

(iii) Finally, evanescent modes associated with purely imaginary wave number give a real value for $\mathcal{J}_n$.

The relation (for right-going modes) $(\mathbf{X}_n|\mathbf{Y}_m) = \mathcal{J}_n \delta_{mn}$ is used to calculate $(\mathbf{X}_n|\overline{\mathbf{Y}}_m)$. Indeed, with $\mathbf{Y}_m = \mathbf{Y}(k_m)$, the latter product is nonzero only when $\overline{k_m} = \pm k_n$. Thus, for real wave number (propagating modes), it is nonzero for $n=m$. For complex k_n with nonzero real part $(k_{n+1} = -\overline{k_n})$, it is nonzero for $m=n+1$. Symmetrically, considering k_{n+1}, it is nonzero for $m=n-1$. Finally, for purely imaginary wave number, it is nonzero again for $n=m$. The structure of the matrix $(\mathbf{X}_n|\overline{\mathbf{Y}}_m)$ is thus the following:

$$\begin{pmatrix} \mathcal{J}_1 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{J}_2 & 0 \\ 0 & \mathcal{J}_3 = \overline{\mathcal{J}_2} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{J}_4 \end{pmatrix}, \tag{C5}$$

where we have considered a propagating mode with wave number k_1 ($\mathcal{J}_1$ is purely imaginary), and three evanescent modes: two are associated with k_2 and $k_3 = -\overline{k_2}$ with nonzero real part and one is associated with a purely imaginary wave number k_4 ($\mathcal{J}_4$ is real).

We can now define the M in (3.8). It contains blocks of identity matrices for real or purely imaginary wavenumbers and blocks of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for two complex conjugate wave numbers. In the previous example, we would have

$$\mathsf{M}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{C6}$$

An important property of M is that $\mathsf{M}^2=I$ with $\bar{\mathsf{M}}=\mathsf{M}$.

[1] E. Gerjuoy and D. S. Saxon, ''Variational principles for the acoustic field,'' Phys. Rev. **94**, 1445–1458 (1954).

[2] T. W. Dawson, ''Scattering matrix and boundary integral equation methods for long range propagation in an acoustic waveguide with repeated boundary deformations,'' J. Acoust. Soc. Am. **90**(3), 1560–1574 (1991).

[3] H. Hudde and U. Letens, ''Scattering matrix of a discontinuity with a nonrigid wall in a lossless circular duct,'' J. Acoust. Soc. Am. **78**(5), 1826–1837 (1985).

[4] V. Varatharajulu and Y.-H. Pao, ''Scattering matrix for elastic waves. I. Theory,'' J. Acoust. Soc. Am. **60**(3), 556–566 (1976).

[5] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics (Non-relativistic Theory)* (Pergamon, Oxford, 1965).

[6] R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966).

[7] P. A. Mello, P. Pereyra, and N. Kumar, ''Macroscopic approach to multichannel disordered conductors,'' Ann. Phys. **181**(2), 290–317 (1988).

[8] C. Vassalo, *Théorie des guides d'ondes Electromagnétiques* (Eyrolles, Paris, 1985), Vol. 2.

[9] R. Carminati, J. J. Sáenz, J.-J. Greffet, and M. Nieto-Vesperinas, ''Reciprocity, unitarity, and time-reversal symmetry of the S-matrix of fields containing evanescent components,'' Phys. Rev. A **62**, 0127121–0127127 (2000).

[10] A. Folguera and G. H. Harris, ''Coupled Rayleigh surface in a slowly varying elastic waveguide,'' Proc. R. Soc. London, Ser. A **455**, 917–931 (1999).

[11] V. Pagneux and A. Maurel, ''Lamb wave propagation in inhomogeneous elastic waveguides,'' Proc. R. Soc. London, Ser. A **458**, 1913–1930 (2002).

[12] P. Morse and U. Ingard, *Theoretical Acoustics* (McGraw Hill, New York, 1952).

[13] J. D. Achenbach, *Wave Propagation in Elastic Solids* (North-Holland, Amsterdam, 1987).

[14] W. B. Fraser, ''Orthogonality relation for the Rayleigh–Lamb modes of vibration of a plate,'' J. Acoust. Soc. Am. **59**, 215–216 (1976).

[15] C. P. A. Wapenaar, ''Reciprocity theorems for two-way and one-way wave vectors: A comparison,'' J. Acoust. Soc. Am. **100**, 3508–3518 (1996).

[16] I. A. Viktorov, *Rayleigh and Lamb Waves: Physical Theory and Applications* (Plenum, New York, 1967), Chap. 2.

[17] P. Roux, B. Roman, and M. Fink, ''Time reversal in an ultrasonic waveguide,'' Appl. Phys. Lett. **70**, 1811–1813 (1997).